Francesco Grimaldi
(1613–1663)

Christiaan Huygens
(1629–1695)

Thomas Young
(1773–1829)

Arnold Sommerfeld
(1868–1951)

Augustin Fresnel
(1788–1827)

Gustav Kirchhoff
(1824–1887)

Henri Poincaré
(1854–1912)

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Arnold Sommerfeld

Mathematical Theory of Diffraction

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Contents

Preface vii

Introduction 1

Mathematical Theory of Diffraction 9

1. General problem formulation 9
2. Expansions in Bessel functions 13
3. Transition from $\Delta u = 0$ to $\Delta u + k^2 u = 0$ 22
4. Bessel functions as the simplest examples 27
5. Everywhere finite solutions 33
6. Solutions with a singularity 40
7. Graphical treatment of the simplest multivalued solution 49
8. Application to diffraction 57

Tafel 67

Translators' Notes 69

References 135

Appendix I: The History and Present State of Discoveries relating to Vision, Light and Colours 141

Appendix II: On the Mathematical Theory of Diffraction Phenomena 151

Index 155

Preface

Traduttore, traditore. If we are traitors, our translation is as faithful to the original as we could make it. The literal rendering of German to English is meant to give an unaltered transmission of both Sommerfeld's ideas and his youthful vigor. Notation and format have been carefully preserved. Liszt may dare to arrange Bach, but most of us had better not. Technical notes and references are added separately. The references are primarily historical. We give a detailed presentation of the contemporary diffraction theories that play a role in Sommerfeld's paper, but do not attempt to trace subsequent advances or applications in a systematic way. Translators' notes are indicated in the text by numerical superscripts. Sommerfeld's own footnotes have been reproduced as they were printed.

It is a pleasure to acknowledge the editors at Birkhäuser Boston for their support in the publication of this somewhat unusual volume of the *Progress in Mathematical Physics* series. We have been fortunate to find the broad-minded view of progress once expressed by Verdi: *torniamo all'antico e sarà un progresso.*

Mathematical Theory of Diffraction

Introduction

Arnold Johannes Wilhelm Sommerfeld, one of the great men of twentieth century physics, was born in Königsberg, Prussia in 1868. An intellectually gifted youth ("In school I was almost more interested in literature and history than in the exact sciences; I was equally good in all subjects, including ancient languages"[1]), he entered the University of Königsberg in 1886. German students at the time generally moved from place to place, but Sommerfeld remained at Königsberg for five years: "The excellence of the professorial chairs in mathematics (Lindemann[2] as *Ordinarius*, Hurwitz[3] as *Extraordinarius*, Hilbert[4] as *Privatdozent*) prevented me from changing universities; this was in a way unfortunate, since I was at the same time a *Burschenschaftler*,[5] and was thus diverted from consistent study." His 1891 doctoral dissertation, *Die willkürlichen Functionen in der mathematischen Physik*, an analysis of series expansions in circular, cylindrical, and spherical functions, was "conceived and written down in a few weeks." The dissertation contains the following vita:

> I, Arnold Sommerfeld, son of the practicing physician Dr. F. Sommerfeld and his wife Cäcilie, born Mathiass, evangelical faith, was born on December 5, 1868 at Königsberg in Prussia. I attended the Altstädtische Gymnasium there, graduated with the

[1] Arnold Sommerfeld, *Autobiographische Skizze*, in *Gesammelte Schriften* (Friedr. Vieweg & Sohn, Braunschweig, 1968, Vol. 4, pp. 673–679). All subsequent unreferenced quotations in the introduction are from this source.

[2] Carl Louis Ferdinand von Lindemann (1852–1939) is most famous for proving that the number π is transcendental (that is, not the root of an algebraic equation). In addition to Sommerfeld, he supervised more than sixty other doctoral students.

[3] Adolf Hurwitz (1859–1919) worked in the fields of abstract algebra and complex function theory. He is known among engineers for the Routh–Hurwitz stability criterion, a method for determining the conditions under which all the roots of a polynomial lie in the left half of the complex plane.

[4] David Hilbert (1862–1943) became one of the most important mathematicians of his time. His definition of twenty-three open problems, presented at the 1900 International Congress of Mathematicians in Paris, has influenced the development of mathematics for more than one hundred years.

[5] Member of a German fraternal society.

A. Sommerfeld et al., *Mathematical Theory of Diffraction*
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> *Zeugnis der Reife*[6] on Michaelmas[7] 1886, matriculated at the local Albertus-University, and devoted myself primarily to the study of mathematics. On July 28 of this year I passed the *examen rigorosum*.
>
> I heard lectures by the following gentlemen:
>
> **Chun, Dahn, Elster, Hilbert, Hurwitz, Lindemann, Lossen, Pape, Rahts, Saalschütz, Thiele, Volkmann, Walter, Wiechert.**[8]
>
> To all these gentlemen, and in particular to Prof. Lindemann, I owe a debt of eternal gratitude.

After passing the customary examination to qualify as a secondary school teacher and fulfilling his military service duty, Sommerfeld traveled in 1893 to Göttingen, where he had arranged through "personal connections" to become an assistant in the Mineralogical Institute of Theodor Liebisch.[9] His true interests, however, remained in mathematics and mathematical physics, and in 1894 he became an assistant to Felix Klein (1849-1925). Sommerfeld's *Habilitationsschrift*, presented in 1895 upon his appointment to the unsalaried academic position of *Privatdozent*, was the *Mathematische Theorie der Diffraction*.[10]

Diffraction is the name used to describe optical phenomena that cannot be explained by geometric ray theory.[11] The first quantitative experimental descriptions of optical diffraction are attributed to the Jesuit Francesco Maria

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[6]Equivalent of the modern German *Abitur* or French *baccalauréat*.

[7]September 29, the feast of St. Michael the Archangel.

[8]Carl Chun (1852–1914), zoologist; Julius Sophus Felix Dahn (1834–1912), writer, poet and historian; Ludwig Hermann Alexander Elster (1856–1935), political economist; Wilhelm Clemens Lossen (1838–1906), chemist; Carl Johannes Wilhelm Theodor Pape (1836–1906), physicist; Johannes Rahts (1854–1933), astronomer and statistician; Louis Saalschütz (1835–1913), mathematician; Günther Thiele (1841–1910), philosopher; Paul Oskar Eduard Volkmann (1856–1938), geophysicist; Julius Guido Wilhelm Hermann Walter (1841–1922), philosopher; Johann Emil Wiechert (1861–1928), physicist.

[9]Theodor Liebisch (1852–1922) was a professor in Königsberg from 1884–1887 and was apparently an acquaintance of Sommerfeld's family. In an 1894 letter to Sommerfeld's mother Cäcilie, Liebisch's wife Adelheid writes that she is concerned with Sommerfeld's well-being; her husband is of the opinion "that Arnold will never be suitable as a mineralogist, that he is a pure mathematician through and through."

[10]*Mathematische Annalen*, Vol. 47, 1896, pp. 317-374. One of the editors of the *Mathematische Annalen* was Felix Klein, which may partly explain how the young Sommerfeld was able to publish a paper of fifty-seven pages. Klein and Sommerfeld later wrote the monumental *Theorie des Kreisels*, which was published in four volumes over the years 1897–1910.

[11]On page 369 of his *Mathematische Theorie der Diffraction*, Sommerfeld defines geometric optics as "that entire way of thinking which decomposes the optical state into individual independently progressing rays."

Grimaldi (1613–1663). Using sunlight entering a darkened room through a small hole as a source of illumination, Grimaldi made careful observations of the shadows cast by opaque bodies. He found that the shadow boundary between light and dark was not sharply defined, and that a series of colored streaks or fringes appeared just outside and parallel to the shadow region. When observing the shadow of a thin rod, Grimaldi found colored fringes inside the shadow. Grimaldi realized that these fringes could not be explained by the known geometric laws of ray propagation, reflection and refraction. He used the Latin word *diffractio* to describe this new and surprising optical phenomenon. Grimaldi's two-volume treatise *Physico-mathesis de lumine, coloribus, et iride* was published posthumously in 1665.[12] The first proposition of the first volume announced the discovery of diffraction in terms that are still used today: "Light is propagated or scattered not only directly and by reflection and refraction, but also in a certain other fourth mode, by diffraction."

Further experimental observations of diffraction phenomena were made in England by Robert Hooke (1635–1703) and Isaac Newton (1642–1727), but no satisfactory theoretical explanation of diffraction was developed.[13] In particular, diffraction phenomena seemed completely incompatible with Newton's corpuscular theory of light. Newton himself gave only the fantastic speculation in his *Opticks*, Book Three, Part I, Query 3: "Are not the Rays of Light in passing by the edges and sides of Bodies, bent several times backwards and forwards, with a motion like that of an Eel? And do not the three Fringes of colour'd Light above-mention'd arise from three such bendings?"[14]

Modern diffraction theory began with the work of the English physician Thomas Young (1773–1829). In a consequential break with the prevailing view of Newton, Young adopted the earlier hypothesis of Christiaan Huygens

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[12] A facsimile of *Physico-mathesis de lumine* was published by the city of Bologna on the tercentenary of Grimaldi's death (Arnaldo Forni Editore, Bologna, 1963).

[13] The experiments of Grimaldi and Hooke are described by the English clergyman and chemist Joseph Priestley (1722–1804) in *The History and Present State of Discoveries relating to Vision, Light and Colours* (London, 1772). Priestley's fascinating and delightful description is presented in Appendix I; the Rare Books and Manuscripts Department at the Boston Public Library is gratefully acknowledged for providing access to an original copy of Priestley's book.

[14] Werner Braunbek and Günther Laukien ("Einzelheiten zur Halbebenen-Beugung," *Optik*, Vol. 9, 1952, pp. 174-179) have used Sommerfeld's diffraction solution to compute the lines of average energy flow that result when a time-harmonic electromagnetic plane wave passes by a perfectly conducting screen. These lines of energy flow do indeed exhibit spatial oscillations near the edge of the screen. Michael Berry ("Exuberant interference: rainbows, tides, edges, (de)coherence ... ," *Philosophical Transactions of the Royal Society of London* A, Vol. 360, 2002, pp. 1023–1037) has generously interpreted these oscillations as a confirmation of Newton's conjecture.

(1629–1695) that light consists of longitudinal wave motion in the luminiferous ether.[15] Young observed, as had Newton before him, that the edge of a diffracting body appears as a luminous line when viewed from within the shadow region.[16] Young was able to account for the fringes that appear near the shadow of a knife edge by computing the interference pattern between the light waves from the primary source and the light waves from the apparent luminous line at the diffracting edge. Although Young had to make an unjustified assumption concerning the phase of the light from the edge in order to make his theoretical results agree with his observations, he made the crucial and ultimately correct step of using the principle of wave interference to explain diffraction phenomena.

Young presented his results to the Royal Society of London in a series of three papers.[17] He was venomously criticized in an anonymous article that appeared in 1803 in the newly founded *Edinburgh Review*.[18] Young wrote a detailed and dignified reply and published it in the form of a pamphlet,[19] of which only one copy was sold. Young's stature as a scientist was seriously damaged, and his work was ignored in England.

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[15]The ether was imagined to be a fluid (and later a solid) which pervaded all space but which somehow did not interfere with the ordinary motion of bodies. The ether retained its place in physical theory throughout the nineteenth century. James Clerk Maxwell (1831–1879), the founder of the modern theory of the electromagnetic field, wrote an interesting article "Ether" for the 1878 *Encyclopædia Britannica*. Sommerfeld "ventured for the first time onto the high seas of theoretical physics" with the 1892 paper "Mechanische Darstellung der elektromagnetischen Erscheinungen in ruhenden Körpern" (*Annalen der Physik*, Vol. 46, pp. 139–151), in which he developed a mechanical interpretation of Maxwell's equations using a "gyrostatic" model of the ether.

[16]Newton's description of the luminous line that appears when the edge of a sharp knife is viewed from the shadow region is given in his *Opticks*, Book Three, Part 1, Observation 5.

[17]Young's papers are reprinted in *Miscellaneous Works of the late Thomas Young* (George Peacock, ed., London, 1855, Vol. 1, articles VII, VIII, and IX).

[18]According to George Peacock in his *Life of Thomas Young, M.D., F.R.S., &c.* (London, 1855, pp. 174–185), the anonymous review was probably written by Henry Peter, Lord Brougham (1778–1868), a sycophantic admirer of Newton who later became Lord Chancellor of England. Peacock gives several excerpts; the following is a disgustingly characteristic example: "It is difficult to deal with an author whose mind is filled with a medium of so fickle and vibratory a nature. Were we to take the trouble of refuting him, he might tell us '*My opinion is changed* and *I have abandoned that hypothesis, but here is another for you.*' We demand, if the world of science which Newton once illuminated, is to be as changeable in its modes as the world of fashion, which is directed by the nod of a silly woman or a pampered fop? Has the Royal Society degraded its publications into bulletins of new and fashionable theories for the ladies of the Royal Institution? *Proh pudor!* Let the Professor continue to amuse his audience with an endless variety of such harmless trifles, but in the name of science, let them not find admittance into that venerable repository which contains the work of Newton, and Boyle, and Cavendish, and Maskelyne, and Herschel."

[19]*Miscellaneous Works*, Vol. 1, article X.

Diffraction theory was developed further in France by Augustin Jean Fresnel (1788–1827). Fresnel's initial work in optics was primarily an independent repetition of what Young had done previously. His first communication to the French Academy in 1815 caught the attention of the astronomer Dominique François Jean Arago (1786–1853). Arago knew of Young's papers and referred Fresnel to them and to Young himself.[20] Fresnel continued his research, and in 1818 submitted a memoir to the Academy in response to a prize competition in diffraction analysis. Fresnel was awarded the prize in 1819, and his memoir contains what is now known as the Fresnel theory of diffraction.[21]

Fresnel based his theory on Huygens' principle, the assertion that each point of a propagating wave front may be regarded as a source of secondary spherical waves. Fresnel used Huygens' principle to compute the far-field interference pattern due to the secondary sources on a wavefront near an edge or in an aperture. By including a somewhat arbitrary inclination factor for the waves emitted by the secondary sources and making the assumption that the diffracting body is a perfect absorber of the incident light, Fresnel found that this interference pattern accurately accounted for the fringes that are observed in diffraction experiments. Fresnel's method can be used to analyze diffraction by straight edges, thin rods, slits, circular and rectangular apertures, and circular disks. Accurate predictions of the fringes near the shadow boundaries are obtained in all these cases.

The Fresnel theory is geometric or kinematic, in the sense that its results are derived not as solutions to a mathematical equation but rather from fundamentally geometric considerations. In 1859, Hermann Ludwig Ferdinand von Helmholtz (1821–1894) gave an analytic basis to Fresnel's theory by deriving Huygens' principle for monochromatic waves in the form of an integral equation. A more general integral equation representation of Huygens' principle, valid for waves of arbitrary time dependence, was derived in 1882 by Gustav Kirchhoff (1824–1887).[22] With significant approximations and assumptions, the application of Kirchhoff's integral equation to diffraction problems gives results that are essentially the same as those of Fresnel. Kirchhoff showed that

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[20] A series of cordial letters between Fresnel and Young is included in Young's *Miscellaneous Works*, Vol. 1, article XVII.

[21] References to the works of Fresnel and a more detailed account of the theories of Fresnel and Young are given by Charles F. Meyer in his very fine *The Diffraction of Light, X-Rays, and Material Particles* (University of Chicago Press, 1934, Chapter 1).

[22] The original papers of Helmholtz and Kirchhoff are cited by B. B. Baker and E. T. Copson in *The Mathematical Theory of Huygens' Principle* (Chelsea Publishing Co., New York, 1987, Chapter I, §4.2 and §5.1).

the success of the Fresnel theory is due in part to the fact that the wavelength of light is very small compared to the dimensions of ordinary diffracting bodies and apertures. If longer wavelengths are considered, the approximations and assumptions of the Fresnel and Kirchhoff theories fail, and the results of the theories are no longer accurate.[23]

Diffraction theory was of practical importance because of its applications to the design and use of optical instruments. It was also of interest to mathematicians since it comprised challenging unsolved problems in field theory. In 1892, Henri Poincaré (1854–1912) published a paper in which he attempted to solve a two-dimensional diffraction problem directly from the differential equation of light propagation, without the use of Kirchhoff's integral equation. Poincaré, one of the greatest mathematicians of the time, derived a remarkable but still approximate solution. In fact, the simplest special case of Poincaré's solution became singular in the shadow boundary region where the Kirchhoff theory gave its most accurate results.[24]

Seen in the context of previous diffraction analyses, Sommerfeld's 1895 *Mathematische Theorie der Diffraction* is strikingly original. Starting from Maxwell's electromagnetic field equations, he first formulates a three-dimensional diffraction problem involving a luminous point and an infinitely thin diffracting body. He then extends the domain of the problem from ordinary space to a Riemann double-space. This allows him to write a simple exact solution of the diffraction problem using what is now called the method of images. The method of images requires the function that describes the field due to a luminous point at a general position on the Riemann double-space. For the two-dimensional problems that Sommerfeld considers in detail, the Riemann double-space reduces to a Riemann surface, and thus Sommerfeld must derive the two-dimensional function that describes the wave field due to a luminous point on a Riemann surface.[25] He does this by systematically de-

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[23]A modern version of the Kirchhoff diffraction theory is presented by the Nobel Laureate Max Born (1882–1970) and Emil Wolf in *Principles of Optics* (Macmillan Co., New York, 1964, Chapter VIII). Kirchhoff's own theory, based on the elastic solid model of the ether, is discussed in translators' note 49.

[24]Poincaré's work is cited and discussed by Sommerfeld in his paper and in translators' note 48.

[25]Sommerfeld had previously considered solutions of the two-dimensional heat conduction equation on a Riemann surface in "Zur analytischen Theorie der Wärmeleitung" (*Mathematische Annalen*, Vol. 45, 1894, pp. 263–277). He writes in his *Autobiographische Skizze* that he was motivated to consider such solutions by the problem of analyzing earth temperature data from an observation station in the Botanical Garden of Königsberg. "The 'Physical-Economic Society,' which my father attended as a member, had in 1891 established a prize for the processing of these data. The station lay near a declivity; thus there arose the problem of investigating heat conduction and integrating the corresponding differential equation in an angular space bounded by two arbitrary half rays inclined toward each other. At that time I had already

veloping a limit process that relates three-dimensional potential functions and two-dimensional wave functions, and obtains the required wave function on a Riemann surface of n sheets in the form of a complex contour integral and in a series expansion. For the simplest case $n = 2$, he transforms the complex contour integral into an integral on the real line. He derives numerically useful far-field approximation formulas for this integral, with careful consideration of the approximation error and the spatial regions in which the approximations are valid, and constructs an extraordinary graphical representation. The diffraction solution in the final section is simply an application of the previously derived results. He gives physical interpretations to the various terms of his solution and shows that the formerly distinct entities of incident, reflected, and diffracted light are completely and smoothly unified. With the bravado of youth and genius, he compares his exact solution to the earlier approximate solutions of Poincaré and Kirchhoff, and "assigns to each its limited region of validity."

The importance of Sommerfeld's work was quickly recognized. The prominent Göttingen physicist Woldemar Voigt (1850–1919) immediately included a discussion of Sommerfeld's result in the second volume of his *Kompendium der theoretischen Physik* (Verlag von Veit & Comp., Leipzig, 1896, V. Teil, IV. Kap., §29). Poincaré[26] referred to Sommerfeld's paper as *"un travail très important"* and described his solution procedure as *"une méthode extrêmement ingénieuse."* A simplified derivation of Sommerfeld's result was published in English by Horatio Scott Carslaw (1870–1954), who spent the summer semester of 1897 as a student of Sommerfeld in Göttingen.[27] All of the

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reformulated the problem by the mirroring method onto a Riemann surface with a winding point, without, however, being able to carry out the integration of the heat conduction equation on such a surface. The work I submitted for the prize contained many unique and, as it seemed to me then, new methods, but was incorrect in an essential point concerning the fulfillment of the boundary conditions, and therefore had to be withdrawn. My work was not continued to numerical treatment, but rather remained mired symbolically in mathematical generalities."

Sommerfeld first published his ideas on the use of Riemann surfaces for the solution of diffraction problems in "Zur mathematischen Theorie der Beugungserscheinungen" (*Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, Nr. 4, 1894, pp. 338–342). A translation of this short paper is given in Appendix II. It is interesting to compare Sommerfeld's initial announcement of his diffraction theory with the full paper published two years later.

[26]H. Poincaré, "Sur la polarisation par diffraction (seconde partie)" (*Acta mathematica*, Vol. 20, 1897, pp. 313–355). This second part of Poincaré's diffraction analysis is discussed briefly at the end of translators' note 48.

[27]H. S. Carslaw, "Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications" (*Proceedings of the London Mathematical Society*, Vol. 30, 1899, pp. 121–161). An additional alternative derivation of Sommerfeld's solution was given by Horace Lamb (1849–1934) in his paper "On Sommerfeld's Diffraction Problem; and on Reflection by a Parabolic Mirror" (*Proceedings*

many subsequent accounts of Sommerfeld's solution are based on this simplified derivation; the more general and more interesting limit process derivation seems to have been completely abandoned. In 1928, Sommerfeld described his original derivation as "very laborious."[28] Laborious, perhaps, for a man of sixty, but there is no sense of labor in the *Mathematische Theorie der Diffraction.* On the contrary, it is thrilling to watch the young lion bare his claws and slash through the difficulties of a problem for which "the full solution is scarcely to be expected."[29]

Sommerfeld went on to a long and illustrious career as a theoretical physicist.[30] We hope that this annotated republication of an important early work will give a first-hand impression of the mind behind that career. The levels of mathematical ability and creativity are wonderfully high. We found a careful reading of Sommerfeld to be enlightening and inspiring. May our inspiration be shared by others who will press forward on his path!

R. J. N.
M. Z.
G. S.

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of the London Mathematical Society, 2nd series, Vol. 4, 1906, pp. 190–203). Lamb writes that his paper may be "a matter of interest to many who do not feel at home in the refined mathematical theories on which Sommerfeld, in his well known paper, has drawn with such effect." Sommerfeld gives a retrospective summary of his diffraction theory in his *Optics* (Volume IV of the *Lectures on Theoretical Physics*, Academic Press, New York, 1954, pp. 247–272).

[28]A. Sommerfeld, "Asymptotische Darstellung von Formeln aus der Beugungstheorie des Lichtes" (*Journal für die reine und angewandte Mathematik*, Vol. 158, 1927/28, pp. 199–208).

[29]John William Strutt, Lord Rayleigh (1842–1919), "Wave Theory of Light," in the 1878 *Encyclopædia Britannica*.

[30]Sommerfeld's life and work are recounted by Paul Forman and Armin Hermann in the *Dictionary of Scientific Biography* (C. G. Gillispie, ed., Charles Scribner's Sons, New York, 1981, Vol. 12, pp. 525–532).

Mathematical Theory of Diffraction.

(with one plate.)

by

A. SOMMERFELD in Göttingen.

The theory of diffraction, as it was founded by Fresnel and made more precise analytically by Kirchhoff, does not satisfy the requirements of mathematical rigor for various reasons. I have already expressed some objections of this type previously†). This theory owes its relatively good agreement with experience merely to the circumstance that the wavelength of light is a very small quantity. For the treatment of Hertzian oscillations and acoustic waves, in which the wavelength is significantly larger, it necessarily proves to be completely unusable. There also exist conditions in optics under which the older diffraction theory is no longer sufficient. In contrast, I give here a mathematically rigorous treatment which is based solely on the differential equations and boundary conditions that have been established in electromagnetic theory. I must, however, limit myself to the very simplest cases, since it seems hopeless from the start to solve the exceptionally complicated problems of ordinary optics in a mathematically satisfactory way. I will speak later of a work by Mr. Poincaré**) that also breaks with the older theory.

§1.
General problem formulation.

We consider an optical state as it is formed by the presence of a luminous point under the influence of a foreign body. We designate the (electrical) components of the light vector

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†) Zur mathematischen Theorie der Beugungserscheinungen. Göttinger Nachr. 1894. Nr. 4.

**) Sur la polarisation par diffraction. Acta Math. Bd. 16.

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by X, Y, Z. Then the differential equations

$$A^2\frac{\partial^2 X}{\partial t^2} = \Delta X, \quad A^2\frac{\partial^2 Y}{\partial t^2} = \Delta Y, \quad A^2\frac{\partial^2 Z}{\partial t^2} = \Delta Z$$

hold. In addition there is the so-called incompressibility condition

$$\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} = 0.$$

We can specialize the dependence on t once and for all. We assume that the state is stationary and periodic, and that the light is monochromatic. Let the period of the light oscillation be τ. Then one may take X, Y, Z in the form

$$X = \mathfrak{Re}\,(e^{\frac{2i\pi t}{\tau}}\,\Xi), \quad Y = \mathfrak{Re}\,(e^{\frac{2i\pi t}{\tau}}\,\mathsf{H}), \quad Z = \mathfrak{Re}\,(e^{\frac{2i\pi t}{\tau}}\,\mathsf{Z}),$$

where Ξ, H, Z are t-independent functions of the spatial coordinates that satisfy the equations

$$\Delta\Xi + k^2\Xi = 0, \quad \Delta\mathsf{H} + k^2\mathsf{H} = 0, \quad \Delta\mathsf{Z} + k^2\mathsf{Z} = 0,$$

$$\frac{\partial\Xi}{\partial x} + \frac{\partial\mathsf{H}}{\partial y} + \frac{\partial\mathsf{Z}}{\partial z} = 0.$$

The constant k introduced here is equal to $\frac{2\pi A}{\tau}$. One calls $\frac{2\pi}{k}$ the wavelength of the light.

In order to obtain additional simple conditions for $\Xi, \mathsf{H}, \mathsf{Z}$, we assume the foreign body to be a plane, infinitely thin, opaque screen. Let the plane of the screen be the x, z plane. We designate the screen by S, and its boundary curve by C. By opaque we mean that the material of the screen is a perfect conductor of electricity. Then the theory of light implies that the (electrical) components of light lying in the plane of the screen vanish at the points of the screen, so that $\Xi = 0$, $\mathsf{Z} = 0$ on S. Furthermore, we will require that the light vector be finite everywhere except at the luminous point P. We wish to assume a luminous point of the simplest type; namely, we assume that the components of light at P become infinite as the Newtonian potential $\frac{1}{r}$ for $r = 0$. We then call P a simple pole. In addition, a requirement concerning infinity is yet to come.

Consequently, the conditions for Z are the following:

a) $\Delta\mathsf{Z} + k^2\mathsf{Z} = 0$ in all space.

b) $\mathsf{Z} = \infty$ as $\frac{1}{r}$ at P, otherwise finite everywhere.

c) $\mathsf{Z} = 0$ in S.

Analogous conditions hold for Ξ. The third component H is determined by the incompressibility condition.

One will try to satisfy such a simple boundary condition as the previous (c) by the *symmetry principle*; one will thus mirror the luminous point P with respect to the plane of S. This would give rise to a second point of infinity in Z at the mirror point P'. The condition b) forbids us, however, to create such a thing anywhere in space.

In ordinary space there is no place for the mirroring process. We therefore construct a doubly-covered space, in which the mirroring process is possible.

We produce such a *Riemann double-space* in the following way. We think of ordinary space as made of two realizations, both of which we cut along the surface S and attach alternatively. The contour curve C forms a branch line for this double-space.[1]

We now consider a function u that is single-valued in the double-space, satisfies the differential equation $\Delta u + k^2 u = 0$, and has a simple pole at the one point P of the double-space. (At the corresponding location $\overline{P}$ in the other realization of space it remains finite.) The function u is determined by the nature of the double-space and by the position of its pole, if one adds a further condition concerning infinity. In order to indicate the dependence of u on the position of its pole, we may write it as $u(P)$.

We now construct the mirror point P' of P with respect to the plane of S; this point now falls into the second realization of space if P lies in the first. If we form the difference

$$v = u(P) - u(P'),$$

then this difference satisfies the conditions posed for Z; for it becomes infinite only at the one point P of the physical space and vanishes on S by symmetry considerations. Therefore, the two-valued function v gives us a solution of our diffraction problem. Furthermore, one notes that v does not vanish simultaneously in the points of the screen plane *outside* C. For only the points *inside* C lie symmetrically between P and P'.[2]

That we speak here of a double-space means, abstractly, nothing other than that we continue the function analytically beyond the region in which it has physical meaning, and that we *consider the physical branch of the function and its analytic continuation at the same time.*

The problem becomes even more interesting mathematically if we specialize it further. Namely, we wish to consider only such states

that are independent of one coordinate (z). Then we must also give the screen a symmetrical form in the direction of z; that is, its boundary curve C extends in straight lines that run parallel to the z-axis. In the same way, we must assume instead of a *luminous point* a *luminous line* parallel to the z-axis. The *Riemann space* then reduces to a *Riemann surface*; of the branch curve C, only its points of intersection with the x, y plane come into consideration, and of the luminous line, only that point in the x, y plane. For this point we will have to prescribe an infinity of the order of the two-dimensional potential $\log \frac{1}{r}$. The differential equation is now

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

The exact solution of the thus specialized diffraction problem leads therefore to the following exercise: *to integrate this well-known differential equation on Riemann surfaces.*

By this statement of the question one is reminded involuntarily of the theory of complex *algebraic functions*, which for the special case $k = 0$ would be classified among the desired functions. Can the methods that lead to the formation of algebraic functions be applied in our case? Directly, this is certainly not possible.

For the equation $\Delta u = 0$ one can, as is well-known, reduce the formation of branched functions to that of single-valued functions, in that one considers instead of the real function u the complex $u + iv = w$, and constructs the symmetric functions of the values of w at the overlying points of the Riemann surface. These symmetric functions become naturally single-valued in the smooth plane and furthermore again satisfy $\Delta u = 0$. Thus, they can be regarded as known. The computation of u then depends only on the solution of an algebraic equation.[3]

For our differential equation, the concept of conjugate function is first of all missing. Here the symmetric functions of the values $u_1, u_2, \ldots, u_n$ of u in overlying points of the Riemann surface are also naturally single-valued in the simple plane, but they are no longer solutions of the same equation. This is the case only for the *symmetric function of the first degree* $u_1 + u_2 + \cdots + u_n$. In the following this can always be regarded as known; namely, it is equal to the solution of the corresponding problem in the smooth plane.

Still, the analogy with the algebraic functions is very useful here as well. Namely, we will discover a way in which one immediately produces from a solution of the differential equation $\Delta u = 0$ with known properties a solution of $\Delta u + k^2 u = 0$ with analogous properties.

A further difficulty arises from the *behavior of our functions at infinity*. Here every solution of our differential equation becomes *essentially singular*. It is therefore not possible to require that our functions be solutions of the differential equation and be continuous on the entire Riemann surface with the inclusion of infinity. Rather, we must imagine the plane (or the Riemann surface) as being delimited by a circle of very large radius, and prescribe boundary values for the points of this circle. From the theoretical standpoint, these boundary values can be assumed arbitrarily. If we wish, however, to arrive at physically useful functions, we will have to choose completely determined boundary values that result from physical considerations.

§2.
Expansions in Bessel functions.

Before we proceed to the construction of special branched functions, we wish to derive some general properties of solutions of our differential equation. It is necessary to consider the *series expansions**) *that correspond to the power series of potential theory.*

1. Let u be a somehow known solution of the differential equation $\Delta u + k^2 u = 0$ that is single-valued and continuous in the neighborhood of the point $x = 0, y = 0$. We describe around this point a circle of radius R, so that in the interior and on the boundary of the circle the function has this same character. We expand the values of u on this circle in a Fourier series with argument $\varphi = \operatorname{artg} \frac{y}{x}$, which is always possible since continuous solutions of our differential equation are necessarily analytic functions**). The series is

$$u = \sum A_m \cos m\varphi + B_m \sin m\varphi.$$

We continue this series in the interior of the circle R in such a way that it satisfies term by term the equation $\Delta u + k^2 u = 0$ transformed into polar

*) These series have already been established in their formal aspects in the repeatedly cited book by F. Pockels: Ueber die partielle Differentialgleichung $\Delta u + k^2 u = 0$, Leipzig 1891. Cf. Theil IV §6.

**) Cf. F. Pockels: l. c. pag. 214.

coordinates, and such that for $r = R$ it continues term by term into the preceding expansion. Then the coefficients of $\begin{matrix}\cos\\ \sin\end{matrix} m\varphi$ must be Bessel functions with argument kr and index m, and, in particular, "Bessel functions of the first kind," since u is finite at $r = 0$ by assumption. Thus we are led to the series

$$(1)\qquad \begin{aligned} u_1 &= \sum J_m(kr)\{\alpha_m \cos m\varphi + \beta_m \sin m\varphi\},\\ \alpha_m &= \frac{A_m}{J_m(kR)}, \qquad \beta_m = \frac{B_m}{J_m(kR)}. \end{aligned}$$

Here it is supposed that none of the quantities $J_m(kR)$ vanishes.

If we had taken u in the preceding to be a solution of the differential equation $\Delta u = 0$, then we would have arrived in exactly the same way at an expansion in increasing powers of r in the form

$$(1')\qquad \begin{aligned} u &= \sum r^m\{\alpha_m \cos m\varphi + \beta_m \sin m\varphi\},\\ \alpha_m &= \frac{A_m}{R^m}, \quad \beta_m = \frac{B_m}{R^m}. \end{aligned}$$

Our series (1) therefore corresponds *formally* to the power series (1′) of potential theory.

2. Let the point $x = 0$, $y = 0$ be an $(n-1)$-fold winding point for the function u. We describe a circle with radius R around this point, so that in the interior and on the boundary of the circle there exist no points of discontinuity and no other winding points besides the origin. On the boundary of this circle we can expand the function in a Fourier series that progresses in multiples of the argument $\frac{\varphi}{n} = \frac{1}{n}\operatorname{artg}\frac{y}{x}$. The series is

$$u = \sum A_m \cos\frac{m}{n}\varphi + B_m \sin\frac{m}{n}\varphi.$$

We continue this series in the interior of the n-fold covered circle according to the condition that it should satisfy the differential equation $\Delta u + k^2 u = 0$ term by term. Then the coefficients of $\begin{matrix}\cos\\ \sin\end{matrix}\frac{m}{n}\varphi$ must be Bessel functions with argument kr and index $\frac{m}{n}$, and, in particular, "Bessel functions of the first kind." We thus arrive at the expansion

$$(2)\qquad \begin{aligned} u_1 &= \sum J_{\frac{m}{n}}(kr)\left\{\alpha_m \cos\frac{m}{n}\varphi + \beta_m \sin\frac{m}{n}\varphi\right\},\\ \alpha_m &= \frac{A_m}{J_{\frac{m}{n}}(kR)}, \quad \beta_m = \frac{B_m}{J_{\frac{m}{n}}(kR)}, \end{aligned}$$

under the repeated assumption that none of the quantities $J_{\frac{m}{n}}(kR)$ vanishes.

For a function u that satisfies the potential equation, the same procedure would have led us to the series

$$(2') \qquad u = \sum r^{\frac{m}{n}} \left\{ \alpha_m \cos \frac{m}{n} \varphi + \beta_m \sin \frac{m}{n} \varphi \right\}, \qquad \alpha_m = \frac{A_m}{R^{\frac{m}{n}}}, \quad \beta_m = \frac{B_m}{R^{\frac{m}{n}}}.$$

The series (2) therefore corresponds, at least *formally*, to the power series (2') of potential theory.

3. The analogy, however, is not only formal: *we can show that the series* (1) *and* (2) *converge exactly to the same extent as the corresponding power series, and that they truly represent the given function* u.

For this purpose we examine the behavior of $J_\nu(x)$ with increasing index ν on the basis of the known series representation

$$J_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \left(1 - \frac{x^2}{2^2(\nu+1)} + \frac{x^4}{2^4 \cdot 2!(\nu+1)(\nu+2)} - \cdots \right).$$

If $\nu + 1 > \frac{x^2}{4}$, then each term of the series becomes smaller than the previous one. Accordingly, it follows that for real positive values of x, under the condition $\nu + 1 > \frac{x^2}{4}$,

$$(3) \qquad J_\nu(x) < \frac{x^\nu}{2^\nu \Gamma(\nu+1)}, \qquad J_\nu(x) > \frac{\vartheta_\nu x^\nu}{2^\nu \Gamma(\nu+1)},$$

where ϑ_ν denotes a nonzero proper fraction with the property that $\vartheta_{\nu'} > \vartheta_\nu$ if $\nu' > \nu$.

Let us then test the convergence of our series, and specifically the series (2), of which the series (1) is only a special case. Let the remainder of the series beginning with a term N be

$$X_N = \sum_N^\infty{}^{(m)} \frac{J_{\frac{m}{n}}(kr)}{J_{\frac{m}{n}}(kR)} \left(A_m \cos \frac{m}{n} \varphi + B_m \sin \frac{m}{n} \varphi \right),$$

where we determine N according to the inequality $\frac{N}{n} + 1 > \left(\frac{kR}{2} \right)^2$, and we wish to suppose that $r < R$. Then there results from (3)

$$|X_N| < \frac{1}{\vartheta_{\frac{N}{n}}} \sum_N^\infty \left(\frac{r}{R} \right)^{\frac{m}{n}} \left| A_m \cos \frac{m}{n} \varphi + B_m \sin \frac{m}{n} \varphi \right|.$$

The quantities A_m and B_m remain beneath a fixed bound, since the original Fourier series converges. It therefore follows that

$$|X_N| < M \sum_N^\infty \left(\frac{r}{R}\right)^{\frac{m}{n}} \quad \text{or also} \quad |X_N| < M' \left(\frac{r}{R}\right)^{\frac{N}{n}},$$

where M and M' denote certain finite numbers that are independent of N. Consequently we can, by taking N even larger if necessary, make the remainder of the series (2) arbitrarily small for all points $r < R$.

On the other hand, the sum of the first N terms of (2) is always a finite quantity if R is not equal to a root of one of the N equations $J_{\frac{m}{n}}(kr) = 0$ $[m < N]$, which we have for the time being excluded. The *convergence* of the series (2) is therefore proven under this assumption.

Furthermore, we must convince ourselves that our series truly *continues into the given values of* u for the limit $r = R$. We have represented the latter through a Fourier series. Now it is known that a Fourier series converges absolutely if the represented function has a continuous first derivative in the interval of the expansion. But this is known for the solutions of our differential equation*). Thus we can, through the choice of a value $m = M$ that moreover satisfies the inequality $\frac{M}{n} + 1 > \left(\frac{kR}{2}\right)^2$, obtain

$$\sum_M^\infty \left| A_m \cos \frac{m}{n} \varphi + B_m \sin \frac{m}{n} \varphi \right| < \varepsilon \vartheta_{\frac{M}{n}},$$

where ϑ may be the proper nonzero ratio that appears in (3). Then for $r \leq R$, the remainder of the series (2) beginning from this term M will satisfy

$$|X_M| < \frac{1}{\vartheta_{\frac{M}{n}}} \sum_M^\infty \left(\frac{r}{R}\right)^{\frac{m}{n}} \left| A_m \cos \frac{m}{n} \varphi + B_m \sin \frac{m}{n} \varphi \right| < \varepsilon.$$

On the other hand, the sum of the first M terms of the series (2) is a continuous function of r that extends continuously into the sum of the first M terms of the Fourier series of u if $r = R$. Thus we have

$$\lim_{r=R} |u - u_1| < \varepsilon + \varepsilon \,.\, \vartheta_{\frac{M}{n}},$$

whereby our proposition is proven.

*) Cf. F. Pockels: l. c. pag. 214.

The functions u and u_1 have the following relation:

1. For $r < R$ they satisfy the equations

$$\Delta u + k^2 u = 0, \quad \Delta u_1 + k^2 u_1 = 0,$$

2. $u = u_1$ for $r = R$.

If the circle with radius R is not an "exceptional region"*) of our differential equation (that is, not such a region for which the boundary value problem becomes undetermined), then we conclude immediately that in general $u = u_1$ for $r \leq R$. In this case it is certainly not true that $J_{\frac{m}{n}}(kR) = 0$, so that our restriction given above does not come into play. In the contrary case, we can still determine an arbitrarily smaller circle with radius R' for which this does not occur. For there exist among the concentric circles with radius $r < R$ only a finite number that represent an exceptional region for our differential equation. Now if we set R' instead of R everywhere in the preceding, then all our propositions are certainly valid for that circle and $J_{\frac{m}{n}}(kR') \neq 0$. We can therefore state the following theorem:

Every solution of our differential equation can be expanded, in the neighborhood of a point $x = a$, $y = b$ for which u is continuous, in a series of the form (1) *or* (2), *depending on whether the function u at the point $x = a$, $y = b$ is unbranched or has an $(n-1)$-fold winding point; this series converges absolutely in the interior of a circle centered at the point $x = a$, $y = b$, and extends to the next discontinuity or branch point.*

4. We consider the values of u on an arbitrary fixed azimuth $\varphi = \varphi_0$ as a function of $r : u = f(r)$, where we wish to assume that the function is unbranched at the point $r = 0$. The series (1) yields a representation of f through Bessel functions that may be written as

$$f(r) = \sum_0^\infty a_m J_m(kr). \tag{4}$$

We compare the character of this representation with the character of the Taylor series for the same function. The Taylor series is characterized so that it gives, truncated at the $(n+1)^{\text{st}}$ term, the function and its first n derivatives correctly at the point $r = 0$. We postulate the same for our series (4).

Series of the form of (4) were first investigated by C. Neumann for complex values of the argument. We take

*) Cf. F. Pockels: l. c. pag. 37 and pag. 222.

from his work the result*), which moreover can also be derived easily from the limiting values of the Bessel functions on page 323, that one constructs the m^{th} derivative of such a series at points in the region of convergence by differentiating the series m times *term by term.*

If we let $r = 0$ in equation (4), then there follows, since $J_n(0) = 0$ for $n > 0$ and $J_0(0) = 1$,

$$f(0) = a_0.$$

Thus, with consideration of the known recursion formulas**) for Bessel functions

$$2\frac{dJ_n(x)}{dx} = J_{n-1}(x) - J_{n+1}(x) \quad [n > 0]$$

$$\frac{dJ_0(x)}{dx} = \qquad - J_1(x),$$

we obtain through termwise differentiation of (4)

$$(5) \quad f'(r) = \frac{k}{2}\{a_1 J_0(kr) + (a_2 - 2a_0)J_1(kr) + (a_3 - a_1)J_2(kr) + (a_4 - a_2)J_3(kr) + \cdots\}.$$

For $r = 0$, it follows from this that

$$f'(0) = \frac{k}{2} a_1.$$

Through repeated differentiation one obtains

$$f''(0) = \left(\frac{k}{2}\right)^2 (a_2 - 2a_0).$$

In general, one recognizes that through n-fold differentiation one obtains an expression for $f^{(n)}(0)$ to which only the first $n + 1$ terms of the series (4) contribute.

On the other hand, we consider the series truncated at the $(n + 1)^{\text{st}}$ term, that is, the "approximating function of n^{th} order"

$$F_n(r) = \sum_0^n a_n J_n(kr).$$

If we calculate for this function $F_n(0)$, $F_n'(0)$, $\ldots$, $F_n^{(n)}(0)$, then we obtain exactly the same expressions in the quantities $a_0, a_1, \ldots, a_n$ as for $f(0)$, $f'(0)$, $\ldots$, $f^{(n)}(0)$. We conclude that $F_n(0) = f(0)$, $\ldots$, $F_n^{(n)}(0) = f^{(n)}(0)$; that is, the given function and the approximating function coincide at the

*) C. Neumann: Theorie der Bessel'schen Functionen, Leipzig 1867, §13.

**) Heine, Handbuch der Kugelfunctionen: 2. Aufl. §61. Gl. 44c).

point $r = 0$ together with their first n derivatives. Geometrically, we may express this as follows: the approximating curve $F_n(r)$ has an n-fold contact with the given curve $f(r)$ at the origin, as is known for the approximating curve of the Taylor series. Thus we have in fact shown that *our expansion with Bessel functions and the Taylor expansion with powers achieve the approximation of the given function in an analogous way, namely through a contact that becomes closer term by term.*

We will later see that our series can be derived directly from the corresponding power series.

5. In addition to series expansions that are valid for *the interior of a circle*, we consider those series that converge when the point lies *outside of a given circle.*

Let u be a solution of our differential equation that may have arbitrary branches or singularities in the interior of a circle with radius R, which for the time being is everywhere continuous and unbranched in the finite region of the exterior of this circle, and may have an $(n-1)$-fold winding point at infinity. We draw the circle with radius R on all sheets of the surface and consider only the values $r > R$. Here we must call upon "*Bessel functions of the second kind.*" By these we mean, in general, solutions of the Bessel equation that become infinite at the origin. In the general case, when the index ν of the Bessel function is not an integer, we can choose the function $J_{-\nu}$. In the case of integer ν, this is identical with $J_{+\nu}$ (except for a factor $(-1)^\nu$). In order to include both cases, we introduce as the "Bessel function of the second kind" a function U_ν that is derived in the following paragraph and which can be defined in terms of a complex integral as

$$U_\nu(x) = \frac{1}{2i} \int e^{ix\cos\alpha} e^{i\nu(\alpha - \frac{\pi}{2})}\, d\alpha, \tag{6}$$

the integration variable α going, for example, from $-\beta + i\infty$ to $+\beta - i\infty$ $[0 < \beta < \pi]$. In the case of an integer ν, this function[4] can be expressed in terms of the functions J_ν and K_ν used by Heine as

$$U_\nu(x) = K_\nu(x) - \frac{i\pi}{2} J_\nu(x). \tag{7}$$

Now we expand the given function u on the circle R in a Fourier series

$$u = \sum A_m \cos\frac{m}{n}\varphi + B_m \sin\frac{m}{n}\varphi.$$

We derive through the assumption

$$\sum_{0}^{\infty}{}^{(m)} U_{\frac{m}{n}}(kr)\left\{\alpha_m \cos\frac{m}{n}\varphi + \beta_m \sin\frac{m}{n}\varphi\right\},$$

(8)
$$\alpha_m = \frac{A_m}{U_{\frac{m}{n}}(kR)}, \quad \beta_m = \frac{B_m}{U_{\frac{m}{n}}(kR)}$$

a series that satisfies the differential equation term by term, and which for $r = R$ continues term by term into the preceding series. To prove the convergence of this series, we need above all inequalities which correspond to the inequalities (3) for $J_\nu(x)$, and which allow us to estimate the behavior of $U_\nu(x)$ with increasing ν.

We make in (6) the substitution $\frac{x}{2}e^{i(\alpha-\frac{\pi}{2})} = z$, from which there follows

$$2U_\nu = \left(\frac{2}{x}\right)^\nu \int e^{-z+\frac{x^2}{4z}} z^{\nu-1}\, dz.$$

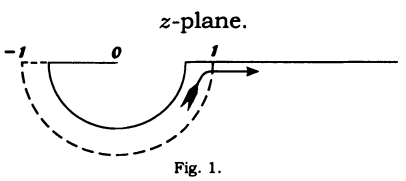

Fig. 1.

The integration path is transformed into a path equivalent to that in the adjacent figure.[5] If we increase ν, then that part of the integral which runs in the interior of the unit circle vanishes. We then have

$$\operatorname*{Lim}_{\nu=\infty} 2\left(\frac{x}{2}\right)^\nu U_\nu = \operatorname*{Lim}_{\nu=\infty} \int_1^\infty e^{-z} e^{\frac{x^2}{4z}} z^{\nu-1}\, dz,$$

or, since all remaining parts of the integrand are positive,

$$\int_1^\infty e^{-z} z^{\nu-1}\, dz < 2\left(\frac{x}{2}\right)^\nu U_\nu < e^{x^2} \int_1^\infty e^{-z} z^{\nu-1}\, dz \quad \text{(in the limit } \nu = \infty\text{)}.$$

The inequality is not changed if we replace the lower limit 1 by 0. Thus we finally obtain, in the limit $\nu = \infty$,

(9)
$$U_\nu > \frac{1}{2}\left(\frac{2}{x}\right)^\nu \Gamma(\nu), \quad U_\nu < \frac{1}{2}\left(\frac{2}{x}\right)^\nu e^{x^2}\Gamma(\nu).$$

More exact consideration shows, further, that one can also give a *finite* number ν such that for all larger values our inequalities (9) are correct.

We now separate from the series (8) a remainder term X_N, where we determine the index position N in such a way that for all values $\nu \geq \frac{N}{n}$ the inequalities (9) hold. Then

$$|X_N| < \sum_N^\infty \frac{|U_{\frac{m}{n}}(kr)|}{|U_{\frac{m}{n}}(kR)|} |A_m \cos\frac{m}{n}\varphi + B_m \sin\frac{m}{n}\varphi| < e^{(kr)^2} M \sum_N^\infty \left(\frac{R}{r}\right)^{\frac{m}{n}},$$

where M denotes the largest value of $|A_m \cos \frac{m}{n}\varphi + B_m \sin \frac{m}{n}\varphi|$, or also, under the assumption $\infty > r > R$,

$$|X_N| < M' \left(\frac{R}{r}\right)^{\frac{N}{n}} .$$

Here M' denotes a number that is independent of N. Therefore, the remainder of the series (8) can be made arbitrarily small through the choice of N for all values $\infty > r > R$.

On the other hand, the sum of the first N terms in (8) is certainly a finite number if it does not happen that one of the quantities $U_{\frac{m}{n}}(kR)$ vanishes. This we can avoid by describing instead of the circle of radius R a somewhat larger one with radius R', for which $U_{\frac{m}{n}}(kR') \neq 0$. The convergence of the series (8) is thus proven for the entire region $\infty > r > R$.

If we had considered the same problem for a solution u of the potential equation, then we would have come in the same way to a series that progresses in negative powers of r and which converges for all values $r > R$. We see:

The series (8) corresponds not only formally, but also with regard to its convergence, to a series that progresses in powers of $\frac{1}{r}$.

The analogy is nevertheless not entirely complete. Namely, in potential theory we can immediately conclude that the series actually represents the function u if we add the hypothesis that u is also continuous at infinity. For our differential equation this is impossible. For the point at infinity is here, as already mentioned, always a singular point. The problem of continuing a function u, which is given only on a circle of radius R, as a solution of the differential equation for values $r > R$, is here entirely ill-posed. One recognizes this already from the following: if we suppose for the moment that there are no branches at infinity, then we could have also written instead of (8) a series that progresses in Heine functions K_m and that continues term by term into the Fourier expansion of u on the circle of radius R. This would also converge for $r > R$, since the inequalities (9) are also valid for the functions K. It would, however, obviously differ from the series (8). *It cannot be said, therefore, that (8) necessarily represents the function u.*

6. To be certain that a series expansion of the given function u is obtained, one must furthermore exclude infinity

through a second circle of radius $R_1 > R$, and construct an expansion for the resulting circular ring which is composed of one series that progresses in the functions J and one series that progresses in the functions U. The coefficients of these series are determined partly through the values of u at $r = R$ and partly through those at $r = R_1$. One then proves that the series with Bessel functions of the first kind converges for all values $r < R_1$, and the series with Bessel functions of the second kind converges for all values $r > R$. The sum of the two series therefore converges in the interior of the circular ring and actually represents the function there. Thus one arrives at the *transfer of the Laurent theorem to the domain of the differential equation* $\Delta u + k^2 u = 0$.

§3.
Transition from $\Delta u = 0$ to $\Delta u + k^2 u = 0$.

While the *method of power series* from function theory remains valid for our differential equations with appropriate modifications, the *method of algebraic calculation of branched solutions*, as discussed in §1, fails completely from the start. However, we will be able to make use of this method indirectly, in that we now give a *procedure through which one can derive from a solution of the equation* $\Delta u = 0$ *with certain properties a solution of the differential equation* $\Delta u + k^2 u = 0$ *with corresponding properties*. Consequently, we can still obtain branched solutions of our differential equation from single-valued solutions through algebraic operations. We do not, however, apply these operations directly to the solution of $\Delta u + k^2 u = 0$, but rather to the corresponding function u in the plane of $\Delta u = 0$, and transfer the so obtained branched potential into the $\Delta u + k^2 u = 0$ plane by our procedure.

We take a detour into the theory of *spherical functions*. We start from an arbitrary complex function $f(\mathsf{Z})$ and relate the Z plane stereographically to a sphere of unit radius. Let the center of the sphere lie at the origin of the Z plane and let it be at the same time the origin of a spatial rectangular coordinate system ξ, η, ζ. Let the ζ-axis be perpendicular to the Z plane, and let the ξ and η axes lie on the real and imaginary axes, respectively, of the Z plane. Let the center of projection be the south pole of the unit sphere, and therefore the point $\xi = 0, \eta = 0, \zeta = -1$. The relation between the Z plane and the unit sphere is, according to these definitions, given by the equation[6]

$$\mathsf{Z} = \frac{\xi + i\eta}{\zeta + 1}, \tag{1}$$

if we also consider the equation of the sphere $\xi^2 + \eta^2 + \zeta^2 = 1$.

The function

$$f\left(\frac{\xi+i\eta}{\zeta+1}\right)$$

is known to be a *two-dimensional potential* on the unit sphere. From it we obtain a function that is defined in all of space and furthermore satisfies the differential equation of the *spatial potential,* if we assign to all points of a given half-ray passing through the center of the sphere the same functional value that exists at the intersection point of the half-ray with the unit sphere. In this way we arrive at the function

$$f\left(\frac{\xi+i\eta}{\zeta+\rho}\right), \qquad \rho^2=\xi^2+\eta^2+\zeta^2. \tag{2}$$

One calls a spatial potential that is a homogeneous function of the rectangular coordinates a *spatial spherical function* (solid spherical harmonic); if one limits oneself to points of the unit sphere, then one speaks of a *spherical surface function* (surface spherical harmonic). In (2) we therefore have a spatial spherical function (of degree 0, since it is homogeneous of the 0^{th} degree).[7] We further obtain a homogeneous function of the -1^{st} degree, which in the same way satisfies the potential equation, if we transform space through the reciprocal radii with respect to the unit sphere. There follows

$$\frac{1}{\rho}f\left(\frac{\xi+i\eta}{\zeta+\rho}\right). \tag{3}$$

We further decrease the degree by 1, 2, ... units by one-fold, two-fold, ... differentiation with respect to one of the coordinates, or more generally through differentiation in an arbitrary "direction"*). Thus we construct very general spherical functions of arbitrarily high order.

This idea comes in a more special form from Maxwell*), who derived the usual rational spherical functions starting from the complex function $f=1$ by appropriately arranging the directions of differentiation.[8] The generalization to an arbitrary function f used here was proposed by Mr. F. Klein**) in a series of lectures on Lamé functions in 1889–90.

In the following, we consistently fix the direction of differentiation as the ζ-axis, which is distinguished by its position with respect to the Z plane. We therefore construct (with the addition of appropriate numerical factors)

$$\frac{(-1)^{m+1}}{m!}\frac{\partial^m}{\partial\zeta^m}\frac{1}{\rho}f\left(\frac{\xi+i\eta}{\zeta+\rho}\right). \tag{4}$$

*) Cf. Maxwell in his Treatise Chap. IX.

**) Cf. F. Pockels l. c. pag. 31.

In order to make the complicated differentiation process more convenient, we transform this expression according to *Cauchy's theorem*. We obtain

$$\frac{1}{2\pi i}\int f\left(\frac{\xi+i\eta}{z+\rho}\right)\frac{1}{(\zeta-z)^{m+1}}\frac{dz}{\rho}, \tag{5}$$

where $\rho^2 = \xi^2+\eta^2+z^2$ is now introduced. The integral is, according to Cauchy's theorem, to be evaluated on a closed path around the point $z=\zeta$ in the plane of the complex variable z, on which one has this point to the left.

We further note that instead of this path we may choose any other arbitrary path that is closed and may not be contracted to a point. Then we still obtain a spherical function of $-m-1$ degree*) (though not necessarily identical to (4)). Further, we can take for m a negative integer or any arbitrary number. In the latter case, we must only take into consideration the branching of the factor $(z-\zeta)^{-m-1}$ from

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*) The proof is based on the following: if we denote the expression (3) by u, there follows the equation

$$\frac{\partial^2 u}{\partial\xi^2}+\frac{\partial^2 u}{\partial\eta^2}+\frac{\partial^2 u}{\partial\zeta^2}=0.$$

If we denote the same expression, where we have substituted z instead of ζ, by u_z, then the following identity is valid for a totally arbitrary integration path (a,b):

$$\int_a^b\left(\frac{\partial^2 u_z}{\partial\xi^2}+\frac{\partial^2 u_z}{\partial\eta^2}+\frac{\partial^2 u_z}{\partial z^2}\right)\frac{dz}{(\zeta-z)^{m+1}}=0. \tag{a}$$

Now

$$\int_a^b\frac{\partial^2 u_z}{\partial z^2}\frac{dz}{(\zeta-z)^{m+1}}=\left[\frac{\partial u_z}{\partial z}\frac{1}{(\zeta-z)^{m+1}}\right]_a^b-\int_a^b\frac{\partial u_z}{\partial z}\frac{(m+1)\,dz}{(\zeta-z)^{m+2}}$$

and

$$-\int_a^b\frac{\partial u_z}{\partial z}\frac{(m+1)\,dz}{(\zeta-z)^{m+2}}=-\left[u_z\frac{m+1}{(\zeta-z)^{m+2}}\right]_a^b+\int_a^b u_z\frac{(m+1)(m+2)\,dz}{(\zeta-z)^{m+3}}.$$

But for the integral on the right side of the last equation we can write

$$\frac{\partial^2}{\partial\zeta^2}\int u_z\frac{dz}{(\zeta-z)^{m+1}}.$$

If the integration path is closed ($a=b$), then the expressions in the brackets [ ] vanish. We therefore conclude from equation (a) that

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\zeta^2}\right)\int u_z\frac{dz}{(\zeta-z)^{m+1}}=0. \tag{b}$$

We note for later use that this equation is also correct for a nonclosed path if the expressions in the brackets [ ] vanish for the limits a and b of the integral. This occurs, for example, if $a=\infty$.

$z = \zeta$ to $z = \infty$, and choose as an integration path a double loop[9] that circles the points $z = \zeta$ and $z = \infty$ in the opposite sense. In this way we can define spherical functions of arbitrary degree that have a simple relation to f.

If the starting function f is branched, then the spherical function will be branched as well. However, the branch structure of the integrand does not correspond exactly to that of f. Namely, we have introduced through the stereographic projection the branch points $\rho = 0$; that is, $z = \pm i\sqrt{\xi^2 + \eta^2}$, which do not belong to the starting function. We can eliminate these afterwards by adding to the integral (5) the following, in which ρ is replaced by $-\rho$:

$$-\frac{1}{2\pi i}\int f\left(\frac{\xi + i\eta}{z - \rho}\right)\frac{1}{(\zeta - z)^{m+1}}\frac{dz}{\rho}. \tag{6}$$

Geometrically, this means that we first mirror the Z plane through the origin (that is, we replace Z by $\mathsf{Z}' = -\mathsf{Z}$), we second project stereographically from the North pole of the unit sphere onto this plane (that is, we set $\mathsf{Z}' = \frac{\xi + i\eta}{1 - \zeta}$) and in the remaining apply all the processes that have been described previously to the function f. If we designate by Z_1 that point which in such a way has been mapped onto the point ξ, η, ζ of the sphere, then we have in analogy to (1) the equation

$$\mathsf{Z}_1 = \frac{\xi + i\eta}{\zeta - 1}. \tag{1'}$$

Now the spherical functions represent for our purpose only an intermediate stage. But we arrive at solutions of our differential equation from spherical surface functions through an appropriate *limit process*. Namely, we let m increase to infinity and at the same time let the point ξ, η, ζ approach the North pole along a meridian of the unit sphere. More precisely expressed, we do the following: we set

$$\xi = \frac{kx}{m}, \quad \eta = \frac{ky}{m}, \quad \xi^2 + \eta^2 + \zeta^2 = 1 \tag{7}$$

and understand that x and y are finite, while m is a quantity that grows to infinity. Then the differential equation of the spherical surface functions is transformed into the differential equation $\Delta u + k^2 u = 0$.[10] The quantities x and y become rectangular coordinates in the plane of this differential equation. In addition, we introduce polar coordinates through the equations

$$r^2 = x^2 + y^2, \quad e^{i\varphi} = \frac{x + iy}{r}. \tag{8}$$

In consequence of (7) there results for ζ the expression

$$\zeta = \left(1 - \frac{k^2(x^2+y^2)}{m^2}\right)^{\frac{1}{2}} = 1 - \frac{k^2 r^2}{2m^2} + \cdots . \tag{9}$$

In order to be able to apply the limit process to our integrals, we make an analogous substitution for z as well. It is advisable to let

$$z = i\sqrt{\xi^2+\eta^2}\cos\alpha = \frac{ikr}{m}\cos\alpha. \tag{10}$$

Then[11]

$$\rho = \frac{kr}{m}\sin\alpha, \quad z \pm \rho = \frac{kr}{m}e^{\mp i\alpha + \frac{i\pi}{2}}, \quad \frac{-dz}{\rho} = i\,d\alpha,$$

$$\frac{\xi+i\eta}{z\pm\rho} = e^{i(\varphi-\frac{\pi}{2}\pm\alpha)}, \quad \lim_{m=\infty}\frac{1}{(\zeta-z)^{m+1}} = e^{ikr\cos\alpha}. \tag{11}$$

From the integral (5) there follows

$$-\frac{1}{2\pi}\int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})e^{ikr\cos\alpha}\,d\alpha \tag{12}$$

and from the sum of the integrals (5) and (6)

$$\frac{1}{2\pi}\int\left\{f(e^{i(\varphi-\alpha-\frac{\pi}{2})}) - f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\right\}e^{ikr\cos\alpha}\,d\alpha. \tag{13}$$

The closed path in the z plane is transformed in the α plane to an integration path whose beginning and end points possibly differ by 2π. If the integration path extends to infinity, it must be determined there in such a way that the integral is meaningful. From expressions (12) and (13), it is now easy to show that the differential equation $\Delta u + k^2 u = 0$ is indeed satisfied with such a determination of the integration path, whereby our limit process is justified after the fact.[12]

2. Our procedure yields for the time being only very *special functions*, namely those *that have branches and points of discontinuity in the finite region only at the point* $x = 0$, $y = 0$. This follows from the fact that any point at a finite distance from the North pole on the unit sphere is thrown unavoidably by our limit process to infinity. Let, for example, Z' be a singular point for our starting function in the complex plane, where we suppose $Z' \neq 0$ and $Z' \neq \infty$. In the projection from the South pole, there is,

according to equations (1), (7), and (9), the relation

$$|\mathsf{Z}| = \frac{kr}{m} \tag{14}$$

between the variable Z and that point r, φ into which Z is transformed through our limit process. To a fixed and nonzero value of Z (for example, $\mathsf{Z} = \mathsf{Z}'$), there corresponds therefore a point whose distance from the origin

$$r = \frac{m|\mathsf{Z}|}{k}$$

grows to infinity through the limit process. On the other hand, for projection from the North pole, there is, according to equations (1′), (7), and (9), the relation

$$|\mathsf{Z}_1| = \frac{m}{2kr} \tag{15}$$

for that point Z_1 of the complex plane that is transformed into the point r, φ. To a fixed and noninfinite value of Z_1 (for example, $\mathsf{Z}_1 = \mathsf{Z}'$), there thus corresponds by this projection a point whose distance from the origin

$$r = \frac{m}{2k|\mathsf{Z}_1|}$$

also grows to infinity through the limit process. From consideration of the geometric relations this follows, moreover, immediately.

In order to keep the singular point in the finite region, we must obviously let its image on the unit sphere, which may have the coordinates ξ', η', ζ', approach the North pole at the same rate as we let m grow. Further, we will also successively lower the degree of the spherical function in relation to this variable triplet ξ', η', ζ', first inverting and then progressively differentiating. In the limit $m = \infty$, we then obtain, instead of the *simple* integrals (12) or (13), a *complex double integral* in which one of the integral signs represents the successive lowering of degree with respect to the triplet ξ, η, ζ, and the other with respect to ξ', η', ζ'. For the sake of brevity, however, the details will be treated separately only in special cases (cf. §6).

§4.
Bessel functions as the simplest examples.

We first derive the Bessel functions through our limit process. We will see that the following functions are thus transformed into each other: the powers in the Z plane, the zonal or tesseral spherical functions on the unit sphere, and the Bessel functions in the plane of the differential

equation $\Delta u + k^2 u = 0$. The results of this paragraph are only partly new and serve mainly as preparation for the following.

1. We begin with the 0^{th} power of the Z-variable and take $f = 1$. From (5) on page 332, we obtain a spherical surface function of the $(-m-1)^{\text{st}}$ degree in the expression

$$\frac{1}{2\pi i}\int \frac{1}{(\zeta - z)^{m+1}}\frac{dz}{\rho}, \quad \rho^2 = \xi^2 + \eta^2 + z^2, \quad \xi^2 + \eta^2 + \zeta^2 = 1.$$

A closed path in the z plane that circles the point $z = \zeta$ is equivalent to a simultaneous circulation around the two points $\rho = 0$ (that is, the points $z = \pm i\sqrt{\xi^2 + \eta^2}$).[13] The substitution (10) on page 334 transforms the expression into

$$\frac{1}{2\pi}\int \frac{d\alpha}{(\zeta - \sqrt{\zeta^2 - 1}\cos\alpha)^{m+1}}.$$

The corresponding integration path in the α plane[14] can be taken along the real axis from 0 to 2π when $\zeta \neq 0$, and otherwise, for example, from C to D (cf. Fig. 2). We arrive here at the usual *zonal spherical functions*, and in particular exactly in the Laplace integral representation*).[15] The limit process $m = \infty$ gives, as is known, a solution of $\Delta u + k^2 u = 0$, namely the *simplest Bessel function*

$$J_0(kr) = \frac{1}{2\pi}\int e^{ikr\cos\alpha}\, d\alpha$$

(integration path, for example, from $\alpha = 0$ to $\alpha = 2\pi$).[16]

2. We take $f = Z^\nu$ and understand by ν an arbitrary positive number. From equation (5) on page 332, this yields the corresponding spherical surface function of the $-m-1^{\text{st}}$ degree

$$\frac{(\xi + i\eta)^\nu}{2\pi i}\int \frac{(z+\rho)^{-\nu}}{(\zeta - z)^{m+1}}\,\frac{dz}{\rho}.$$

The branch points of the integrand are the points $z = \pm i\sqrt{\xi^2 + \eta^2}$, for which $\rho = 0$, and the point (or more correctly the points) $z = \infty$, for which $z + \rho = \infty$ or $z + \rho = 0$, depending on whether ρ is calculated with a positive or negative sign. The integration path, which we originally think of as a circulation around $z = \zeta$, may in the general case (when ν is not an integer) be extended only to but not across the point $z = \infty$.[17] If we make the substitution (10) on page 334, there results

*) Cf. Heine, l. c. §9, Gl. 5a.

(1) $$\frac{1}{2\pi}\left(\frac{\xi+i\eta}{\sqrt{\xi^2+\eta^2}}\right)^{\nu}\int\frac{e^{i\nu(\alpha-\frac{\pi}{2})}\,d\alpha}{(\zeta-\sqrt{\zeta^2-1}\cos\alpha)^{m+1}}$$

and after the execution of the limit process

(2) $$\frac{e^{i\nu\varphi}}{2\pi}\int e^{ikr\cos\alpha}e^{i\nu(\alpha-\frac{\pi}{2})}\,d\alpha.$$

The integration path described above is transformed in the α plane, for example, into the path (A). The hatched regions I, II, III in the figure are formed by those locations where $i\cos\alpha$ has a negative real part. If we determine the path (A) so that its infinitely distant parts lie in regions I and II, we ensure that the expression (2) is finite for all positive and real kr.

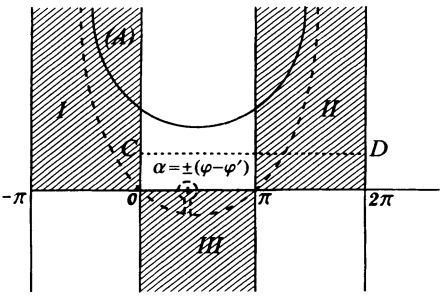

Fig. 2.

The expression (1) is a *tesseral spherical function*, and the expression (2) becomes directly equal to the solution

$$e^{i\nu\varphi}J_\nu(kr)$$

of our differential equation. The *Bessel function* J_ν appears here in an integral representation that is well known*) for the special case of an integer ν, where the path (A) can be reduced to the segment 0 to 2π.

We remark further that when we project from the North pole, where we wish to substitute Z_1 for Z as above, the limit process starting with $Z_1^{-\nu}$ (ν is an arbitrary positive number) produces, as a consequence of equation (6) on page 333, the expression[18]

$$-e^{-i\nu\varphi}\int e^{ikr\cos\alpha}e^{i\nu(\alpha+\frac{\pi}{2})}\,d\alpha=-e^{-i\nu\varphi}e^{i\nu\pi}J_\nu(kr).$$

3. $f=\lg Z$. According equation (5), a corresponding spherical function of the $(-m-1)^{\text{st}}$ degree is

$$\frac{1}{2\pi i}\int\lg\left(\frac{\xi+i\eta}{z+\rho}\right)\frac{1}{(\zeta-z)^{m+1}}\,\frac{dz}{\rho}.$$

The branch points of the integrand are the same as in 2. We wish to connect the points $z+\rho=0$ and $z+\rho=\infty$ by a branch cut, and choose as an integration path a curve that embraces this cut closely on both sides. The logarithm

*) Heine, l. c. Gl. (43, a).

differs on the two parallel banks by the additive quantity $2\pi i$. The integral over the closed path thus reduces to $2\pi i$ multiplied by an integral that is taken simply from $z + \rho = \infty$ to $z + \rho = 0$, while the logarithm under the integral sign falls away.[19] In the transformation (10) on page 334, the limits $z + \rho = \infty$ and $z + \rho = 0$ are transformed to $\alpha = a + i\infty$ and $\alpha = a' - i\infty$, respectively, where a and a' are undetermined.

In order that the integral is also defined after the limit process, we determine the quantities a and a' in such a way that $a + i\infty$ lies in region I and $a' - i\infty$

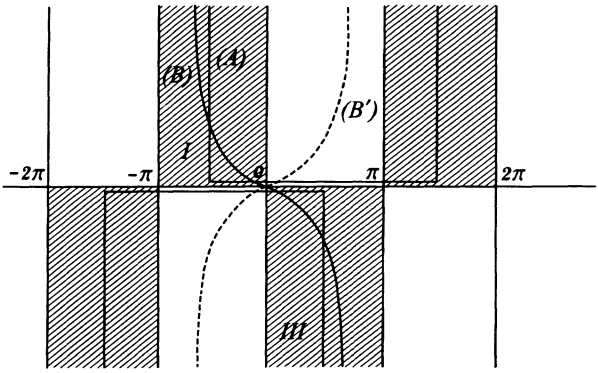

Fig. 3.

lies in region III. We choose, for example, $a = -a' = -\beta$, $0 < \beta < \pi$. We wish to denote the integration path that is determined in this way as path (B), and we further wish to introduce the numerical factor $\frac{1}{2i}$ into the integral over (B). Our spherical function of the $(-m-1)^{\text{st}}$ degree is then

$$\frac{1}{2i}\int \frac{d\alpha}{(\zeta - \sqrt{\zeta^2 - 1}\cos\alpha)^{m+1}};$$

the limit process yields the integral

$$U_0(kr) = \frac{1}{2i}\int d\alpha\, e^{ikr\cos\alpha} \tag{3}$$

taken over the path (B).

4. $f = \lg \mathsf{Z}\,.\,\mathsf{Z}^n$. n denotes an integer. If we construct the corresponding spherical function, there follow the same branches in the z plane as in (3) for the case $n = 0$. We therefore choose the same integration path as there, and arrive through our limit process at the product

$$e^{in\varphi}U_n(kr),$$

where

$$U_n(x) = \frac{1}{2i}\int e^{ix\cos\alpha}e^{in(\alpha-\frac{\pi}{2})}\,d\alpha \quad \text{(integration path } (B)\text{)}. \tag{4}$$

We designate the expressions (3) and (4) as *Bessel functions of the second kind.* They are, as is evident from the integral representation, always finite for real positive x and become ∞ for $x = 0$. In order to determine the type of singularity, the following observation is useful.

The Bessel function of the first kind $J_\nu(x)$ and that Bessel function which results through the substitution of ν by $-\nu$ are expressed according to (2) in the following way:

$$2\pi e^{\frac{i\nu\pi}{2}} J_\nu(x) = \int e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha, \tag{5a}$$

$$2\pi e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x) = \int e^{ix\cos\alpha} e^{-i\nu\alpha}\, d\alpha. \tag{5b}$$

The integration path is the path (A). In particular, we can shape it in such a way that it consists of the straight segments connecting the points

$$-\frac{\pi}{2} + i\infty, \quad -\frac{\pi}{2}, \quad \frac{3\pi}{2}, \quad \frac{3\pi}{2} + i\infty.$$

If we exchange $+\alpha$ with $-\alpha$ in equation (5b), then the path given above becomes

$$-i\infty - \frac{3\pi}{2}, \quad -\frac{3\pi}{2}, \quad +\frac{\pi}{2}, \quad +\frac{\pi}{2} - i\infty.$$

The difference of the right-hand sides of (5a) and (5b) is (cf. Fig. 3) identical with the difference of the integrals over the paths

$$+\frac{\pi}{2} - i\infty, \quad +\frac{\pi}{2}, \quad +\frac{3\pi}{2}, \quad +\frac{3\pi}{2} + i\infty$$

and

$$-\frac{3\pi}{2} - i\infty, \quad -\frac{3\pi}{2}, \quad -\frac{\pi}{2}, \quad -\frac{\pi}{2} + i\infty.$$

Thus we have[20]

$$e^{\frac{i\nu\pi}{2}} J_\nu(x) - e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x) = \frac{(e^{-i\nu\pi} - e^{+i\pi\nu})}{2\pi} \int e^{-ix\cos\alpha} e^{i\nu\alpha}\, d\alpha,$$

where the integral is to be taken over the path (B), reflected by the imaginary axis (the path B' in Fig. 3).

Finally, through the exchange of $-i$ with $+i$ there results the integral[21]

$$\frac{1}{2i} \int e^{ix\cos\alpha} e^{i\nu(\alpha - \frac{\pi}{2})}\, d\alpha = \pi \frac{J_{-\nu}(x) - e^{-i\nu\pi} J_\nu(x)}{2\sin\nu\pi} \tag{6}$$

taken directly over (B). In the case of an integer $\nu = n$, the left-hand side is directly equal to the previously defined function $U_n(x)$, while the numerator

and denominator of the right-hand side vanish. If we conceive of the expression on the left in this case as the limiting value of the same expression for $\nu = n+\varepsilon$ with decreasing ε, then we can deduce its value through differentiation*). One finds[22]

$$U_n(x) = K_n(x) - \frac{i\pi}{2} J_n(x), \tag{7}$$

where $K_n(x)$ is the quantity denoted by Heine as the Bessel function of the second kind, for which the type of singularity at $x = 0$ has been investigated. In particular, $U_0(x)$ becomes infinite as $\log \frac{1}{x}$ for $x = 0$, or in symbols

$$U_0(x) = \lg \frac{1}{x} + \cdots \quad \text{for } x = 0. \tag{8}$$

One should further note that this derivation of a second solution U_n of Bessel's differential equation from the two coinciding solutions J_n and J_{-n} corresponds exactly to the derivation that is customary in the theory of linear differential equations for the case of integer difference of exponents.

If we start with $\mathsf{Z}_1^{-n} \lg \mathsf{Z}_1$ instead of $\mathsf{Z}^n \lg \mathsf{Z}$, in which we imagine the complex plane as being projected from the *North pole,* then we obtain through the same procedure the solution of our differential equation

$$-e^{-in\varphi} e^{in\pi} U_n(kr).$$

5. For the sake of uniformity, we wish to introduce the integral $U_\nu(x)$ as the Bessel function of the second kind also *in the case of an arbitrary* ν, as it was previously used in §2 under the same designation. In the case of an integer ν, we saw that $e^{i\nu\varphi} U_\nu(kr)$ results from the complex function $\lg \mathsf{Z} \,.\, \mathsf{Z}^\nu$ through integration in the plane of the z-variable over a closed path that enclosed the points $z+\rho = 0$ and $z+\rho = \infty$. We could have said more simply: the product $e^{i\nu\varphi} U_\nu(kr)$ results from the power Z^ν if we use as the integration path not that *closed* path, but rather a *simple* curve from the point $z + \rho = 0$ to the point $z + \rho = \infty$. This curve plays for us, according the footnote on page 332, the role of a closed curve, because the terms that appear there in square brackets [ ] vanish for $z + \rho = 0$ and $z + \rho = \infty$. It is transformed in the α plane to the integration path (B). If we take this path as a basis, then our above result is valid for arbitrary ν. Namely, the solution $e^{i\nu\varphi} U_\nu(kr)$ of our differential equation always results from the power Z^ν on the path (B).

6. We bring together the results of this paragraph for easy reference in the following way.

*) Cf. Hankel: Cylinderfunctionen erster und zweiter Art, §1, Math. Ann. Bd. I.

In the plane of the variable α we have used two different integration paths, *a path* (A) that begins at infinity in region I and ends at infinity in region II, and *a path* (B) that begins at infinity in region I and ends at infinity in region III. On the path (A) we introduce into the integral the factor $\frac{1}{2\pi}$, and on path (B) the factor $\frac{1}{2i}$. *Through our procedure there now results*

	on the path (A)	*on the path* (B)
from Z^ν	$e^{i\nu\varphi}J_\nu(kr)$	$e^{i\nu\varphi}U_\nu(kr)$

by projection from the South pole; and there results

from $\mathsf{Z}_1^{-\nu}$	$-e^{i\nu\pi}e^{-i\nu\varphi}J_\nu(kr)$	$-e^{i\nu\pi}e^{-i\nu\varphi}U_\nu(kr)$

by projection from the North pole of the unit sphere.

§5.
Everywhere finite solutions.

In the preceding paragraph, we have constructed solutions of our differential equation for which we were interested essentially in the dependence on *one variable* (r), while the dependence on the other variable (φ) was that of the trigonometric functions. These solutions were either *single-valued in the smooth plane* (ν an integer), or *on a Riemann surface that has an* $(n-1)$*-fold winding point at* $r = 0$ *and at infinity* (ν a rational fraction with denominator n), or *on a winding surface of infinitely high order* (ν irrational). In this paragraph, we will derive solutions that depend essentially on *two variables*, and which are single-valued either *in the smooth plane*, or *on the previously mentioned n-sheeted Riemann surface*, or *on a winding surface of infinitely high order*. For the time being, these shall be *functions that are everywhere finite.*

1. We start from the simple linear function of Z that becomes infinite at one point Z′ of the unit circle, that is 1 at the origin and is 0 at infinity; that is,

$$f = \frac{1}{1 - \frac{\mathsf{Z}}{\mathsf{Z}'}}. \tag{1}$$

We obtain from this, according to the procedure given in §3 section 1, an everywhere finite and single-valued solution in the smooth plane. We have only to insert the special function above into equation (13) on page 334. We designate the resulting function by u_0:[23]

$$(2)\quad u_0 = \frac{1}{2\pi}\int\left\{\frac{1}{1-e^{i(\varphi-\varphi'-\alpha)}}-\frac{1}{1-e^{i(\varphi-\varphi'+\alpha)}}\right\}e^{ikr\cos\alpha}\,d\alpha$$
$$= \frac{1}{2\pi i}\int\frac{\sin\alpha\, e^{ikr\cos\alpha}\,d\alpha}{\cos\alpha-\cos(\varphi-\varphi')}.$$

Here we have set

$$(3)\qquad \mathsf{Z}' = e^{i\varphi'-\frac{i\pi}{2}}\quad(\varphi' \text{ real}).$$

The integral is to be taken over the path (A) in the α plane.

The real axis, and therefore in particular the points $\alpha = \pm(\varphi-\varphi')$, should lie outside the region enclosed by (A). However, if we consider $\cos\alpha$ as the variable of integration, then the point $\cos\alpha = \cos(\varphi-\varphi')$ lies inside the corresponding path in the $\cos\alpha$ plane.[24] Furthermore, a single-valued function of $\cos\alpha$ appears under the integral sign. The integration path can therefore be contracted to the point $\cos\alpha = \cos(\varphi-\varphi')$. According to Cauchy's theorem, we obtain

$$(4)\qquad u_0 = e^{ikr\cos(\varphi-\varphi')}$$

or in rectangular coordinates x, y,

$$(4')\qquad u_0 = e^{ik(x\cos\varphi'+y\sin\varphi')}.$$

The simple function that we thus obtain is that solution of our differential equation with which one operates almost exclusively in ordinary optics. It possesses the following properties: *it is single-valued in the smooth plane and is continuous and finite for all finite values* x, y. Instead of the *exceptional point* $\mathsf{Z} = \mathsf{Z}'$ in the complex plane, we have here only an *exceptional direction* $\varphi = \varphi'$ ("direction of the incident wave"). The function u_0 represents the *undisturbed distribution of parallel light in the infinite plane.*

2. We now seek the solution of our differential equation that is analogous to u_0 on the previously described Riemann surface with an $(n-1)$-fold winding point. For this purpose, we start not from the function (1) that is *single-valued* in the smooth plane, but rather from the *analogous function on our Riemann surface.* This function results very simply from a *conformal mapping.* We need only replace Z by $\mathsf{Z}^{\frac{1}{n}}$ in (1). For convenience, we may also write $\mathsf{Z}'^{\frac{1}{n}}$ instead of Z' and introduce the factor $\frac{1}{n}$. We thus use the starting function

(a) $$f = \frac{\frac{1}{n}}{1 - \left(\frac{Z}{Z'}\right)^{\frac{1}{n}}}.$$

According to equation (13) on page 334, we obtain as the corresponding solution of our differential equation[25]

$$(5)\quad u = \frac{1}{2\pi n}\int \left\{ \frac{1}{1 - e^{\frac{i}{n}(\varphi - \varphi' - \alpha)}} - \frac{1}{1 - e^{\frac{i}{n}(\varphi - \varphi' + \alpha)}} \right\} e^{ikr\cos\alpha}\, d\alpha$$

$$= \frac{1}{2\pi i n}\int \frac{\sin\frac{\alpha}{n}\, e^{ikr\cos\alpha}\, d\alpha}{\cos\frac{\alpha}{n} - \cos\left(\frac{\varphi - \varphi'}{n}\right)}.$$

The integral is to be taken along the path (A); the points $\alpha = \pm(\varphi - \varphi')$ lie outside the region enclosed by the integration path. One sees immediately the following properties of this function:

a) *It satisfies our differential equation* (as do all functions that result from equation (13) through the substitution of an arbitrary complex function f).

b) *It is single-valued on the n-sheeted Riemann surface.*

c) *It is finite and continuous for all finite real values of* r. We examine

d) *its behavior at infinity.* We wish to designate as the *first sheet* of the Riemann surface the collection of those points for which $|\varphi - \varphi'| < \pi$, and as the 2^{nd}, 3^{rd}, ... n^{th} sheet the collection of all other points. We then assert:

At infinity on the first sheet $u = u_0$, *and at infinity on all other sheets* $u = 0$.

For a proof, we consider Figure 2. If $|\varphi - \varphi'| > \pi$, then there is no singular point of the integral on the segment from 0 to π of the real axis. Thus we can extend the path (A) across the real axis by taking it from region I through the point 0 to region III, and through the point π to region II. It then runs only in the hatched region, in which $e^{ikr\cos\alpha}$ vanishes with increasing r. In this case the $\lim_{r=\infty} u$ also vanishes.

If, however, $|\varphi - \varphi'| < \pi$, then when we deform the integration path in an analogous way, we must place a loop around that one of the points $\alpha = \pm(\varphi - \varphi')$ that falls between 0 and π. The integral over this loop gives, according to Cauchy's theorem, exactly u_0, and the integral over the remaining part of the integration path

vanishes in the limit $r = \infty$. In this case, therefore, the function u is transformed with increasing r into u_0. Our assertion is therefore proven.

We can give this result the following *intuitive interpretation.* We first imagine n smooth planes overlying each other. In the first plane we impose the solution u_0 of our differential equation, and on all other planes the solution 0. If we now cut these planes along a half-ray from the origin and join them alternatively to yield our Riemann surface, then there results a disturbance that changes the original state at all finite points. The disturbance does not extend to infinity, however. Here we have continued $u = u_0$ and $u = 0$.

Furthermore, there is the following noteworthy relation between the functions u and u_0:

e) The symmetric function $u_1 + u_2 + \cdots + u_n$ of the values of u "on the overlying points" of the 1^{st}, 2^{nd}, ... n^{th} sheets is exactly equal to u_0. The overlying points are the following: r, φ; $r, \varphi + 2\pi$; ... $r, \varphi + 2(n-1)\pi$. The symmetric function is therefore

$$(6) \qquad u_1 + u_2 + \cdots + u_n = \frac{1}{2\pi i} \int e^{ikr\cos\alpha} \sin\frac{\alpha}{n} \frac{d\alpha}{n} \left\{ \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n}} + \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2\pi}{n}} + \cdots + \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2(n-1)\pi}{n}} \right\}.$$

Now from the identity[26]

$$\cos\alpha - \cos(\varphi - \varphi') = 2^{n-1} \left(\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n}\right)\left(\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2\pi}{n}\right) \cdots \left(\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2(n-1)\pi}{n}\right),$$

one derives through logarithmic differentiation

$$\frac{\sin\alpha}{\cos\alpha - \cos(\varphi - \varphi')} = \frac{1}{n}\sin\frac{\alpha}{n} \left\{ \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n}} + \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2\pi}{n}} + \cdots + \frac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi' + 2(n-1)\pi}{n}} \right\}.$$

Thus the right-hand side of equation (6) is transformed directly into the expression for u_0 in equation (2) — in accordance with a general remark on page 320.

The function u, we can say, represents *the motion of a light wave that is incident on our Riemann surface in the direction* $\varphi = \varphi'$. It must take the place of u_0 in all those two-dimensional optical problems that by their nature give rise to a function that is single-valued not on a smooth plane, but rather on our n-sheeted Riemann surface; namely, in the problems of diffraction by a wedge-shaped opaque body with plane boundary surfaces.[27]

3. According to §2, it is possible to expand every solution of $\Delta u + k^2 u = 0$ that is finite and continuous in the interior of a circle of radius R in a series that progresses in terms of the functions $J_{\frac{m}{n}}(kr) \begin{matrix}\cos \\ \sin\end{matrix} \frac{m}{n}\varphi$, in the case where the origin is an $(n-1)$-fold winding point. Since our functions u and u_0 satisfy this condition for all values $r < \infty$, it must be possible to represent them by a *convergent series in the entire plane* (with the exclusion of infinity). Such series exhibit a strong analogy with the power series of potential theory; *we now derive these directly from the power series of our starting function.*

If we wish to use for this purpose the results of §4, then we must distinguish the projection from the South pole and the projection from the North pole. In the middle expression of equation (5), the second term results by projection from the South pole with the function

$$f(\mathsf{Z}) = \frac{\frac{1}{n}}{1 - \left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{1}{n}}},$$

where Z denotes the point that is transformed into a specified point r, φ, and the first term results from the projection from the North pole with the same function

$$f(\mathsf{Z}_1) = \frac{\frac{1}{n}}{1 - \left(\frac{\mathsf{Z}_1}{\mathsf{Z}'}\right)^{\frac{1}{n}}},$$

where Z_1 is the point that is transformed into the same point r, φ by this projection. According to equations (14) and (15) of page 335 and according to equation (3), we have:

$$|\mathsf{Z}| = \frac{kr}{m}, \quad |\mathsf{Z}_1| = \frac{m}{2kr}, \quad |\mathsf{Z}'| = 1.$$

Through the limit process, therefore, there certainly follows $|\mathsf{Z}| < |\mathsf{Z}'|$, $|\mathsf{Z}_1| > |\mathsf{Z}'|$. Thus

we must expand $f(Z)$ in the complex plane in increasing powers of $\left(\frac{Z}{Z'}\right)^{\frac{1}{n}}$, and expand $f(Z_1)$ in decreasing powers of $\left(\frac{Z_1}{Z'}\right)^{\frac{1}{n}}$.

There follows

$$(7) \qquad f(Z)+f(Z_1)=\frac{1}{n}\Big\{1+\Big(\frac{Z}{Z'}\Big)^{\frac{1}{n}}+\Big(\frac{Z}{Z'}\Big)^{\frac{2}{n}}+\cdots \cdots-\Big(\frac{Z'}{Z_1}\Big)^{\frac{1}{n}}\Big(1+\Big(\frac{Z'}{Z_1}\Big)^{\frac{1}{n}}+\cdots\Big)\Big\}.$$

We wish to integrate this equation on the right and the left along the path (A)*). Then there results on the left, as shown in this paragraph, the function u. If we perform the integration on the right term by term, then we can extract the result from the summary at the end of §4. There follows from

$$\Big(\frac{Z}{Z'}\Big)^{\frac{m}{n}}-\Big(\frac{Z'}{Z_1}\Big)^{\frac{m}{n}} \;\Big|\; e^{i\frac{m}{n}(\varphi-\varphi')}\cdot e^{\frac{im}{n}\frac{\pi}{2}}J_{\frac{m}{n}}(kr)+e^{i\frac{m}{n}(\varphi'-\varphi)}\cdot e^{\frac{im}{n}\frac{\pi}{2}}J_{\frac{m}{n}}(kr)$$

$$=2e^{\frac{im}{n}\frac{\pi}{2}}\cos\frac{m}{n}(\varphi-\varphi')J_{\frac{m}{n}}(kr),$$

while from the constant term 1 there results directly $J_0(kr)$. Thus we find for our function u the uniformly convergent series

$$u=\sum_0^\infty \gamma_m e^{\frac{im}{n}\frac{\pi}{2}}\cos\frac{m}{n}(\varphi-\varphi')J_{\frac{m}{n}}(kr) \qquad \left\{\begin{array}{l}\gamma_m=\frac{2}{n}\ \text{for}\ m>0\\ \gamma_m=\frac{1}{n}\ \text{for}\ m=0\end{array}\right\}.$$

For the simplest case $n=1$ treated in section 1, this series appears in Heine**) and serves there directly as the definition of the Bessel functions.

4. Now it will also be easy to give the corresponding function *for a winding surface of infinitely high order.* We first map this winding surface conformally onto the Z plane, therefore replacing Z by $\lg Z$ and also Z′ by $\lg Z'$ in the function (1). Accordingly, we now place the function

$$f=\frac{1}{1-\dfrac{\lg Z}{\lg Z'}}$$

*) By this abbreviated manner of expression, we mean that we carry out for the left and right sides of equation (7) all the specified operations of our general procedure, and that we choose the path (A) as the integration path for the resulting function of α.

**) l. c. equation (14b).

into equation (13) on page 334. Then there results through integration on the path (A)[28]

$$u' = \frac{1}{2\pi i} \int \frac{2\alpha}{(\varphi - \varphi')^2 - \alpha^2} e^{ikr\cos\alpha}\, d\alpha,$$

an expression that can also be derived from (5) through the limit process $n = \infty$.[29]

From this function one proves, as in section 2, the properties a), c), and d). But instead of b) there appears here:

b′) *The function is single-valued only on a winding surface of infinitely many sheets.*

Finally, we can assemble from this function u', which is *completely aperiodic in* φ, the above functions *of period* 2π *or* $2\pi n$, and therefore arrive at a transference and extension of property e).

Namely, if we form the sum of the values of u' at all the points lying over each other on the winding surface (that is, the sum from $k = -\infty$ to $k = +\infty$ of the values of u' at the points $r, \varphi + 2kn\pi$), then there results a single valued function u on the n-sheeted surface, and in particular the function u_0 for $n = 1$. In fact, we find by logarithmic differentiation of the identity[30]

$$\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n} = \frac{(\varphi - \varphi')^2 - \alpha^2}{2n^2} \prod{}' \frac{\alpha^2 - (\varphi - \varphi' + 2kn\pi)^2}{(2kn\pi)^2}$$

$$\left[\begin{array}{c} k = -\infty \cdots -1 \\ +1 \cdots +\infty \end{array}\right]$$

a relation between

$$\frac{1}{n}\sin\frac{\alpha}{n} : \left(\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n}\right)$$

and a convergent infinite series through which the previously mentioned sum is transformed directly into expression (5) or (2) for u or u_0, respectively.

The function u' represents, so to speak, the *motion of a light wave that is incident on our infinitely-sheeted winding surface in the direction* $\varphi = \varphi'$. We can correspondingly state the above result in the following way:

If we let infinitely many light rays, whose incident directions differ by $2n\pi$, fall onto our infinitely-sheeted winding surface, then the resulting motion of light becomes identical with the motion of a light ray on an n-sheeted winding surface, or (in the case $n = 1$) with the undisturbed motion of a light ray in the smooth plane.[31]

§6.
Solutions with a singularity.

In this paragraph we derive solutions of our differential equation that become infinite at a prescribed point x', y' as $\lg \frac{1}{R}$ for $R = 0$, where R denotes the distance of the variable point x, y from the point x', y'. We call such a singularity a *pole*, and in particular a simple pole. We derive these solutions, according to the method sketched in §3 section 2, from the same function f from which the everywhere finite solutions flowed in the previous paragraph. Solutions with multiple separate or coincident poles can be obtained afterwards from these simplest solutions through summation or differentiation of the pole with respect to the coordinates.

1. The single-valued solution in the smooth plane results from the complex function

$$f = \frac{1}{1 - \frac{\mathsf{Z}}{\mathsf{Z}'}}. \tag{1}$$

We cover the complex plane through stereographic projection from the South pole of the unit sphere. If ξ, η, ζ and ξ', η', ζ' are the coordinates of the image points Z and Z', respectively, then we have

$$\mathsf{Z} = \frac{\xi + i\eta}{\zeta + 1}, \quad \mathsf{Z}' = \frac{\xi' + i\eta'}{\zeta' + 1}. \tag{2}$$

From the two-dimensional potential on the surface of the sphere there follows a spatial spherical function, if we replace 1 in expressions (2) by ρ and ρ', where

$$\rho^2 = \xi^2 + \eta^2 + \zeta^2, \quad \rho'^2 = \xi'^2 + \eta'^2 + \zeta'^2.$$

Through inversion with respect to both triplets of variables, there results a spherical function $\frac{1}{\rho\rho'} f$ that is homogeneous of the -1^{st} degree in ξ, η, ζ *as well as* in ξ', η', ζ'. The degree in both variable triplets is now reduced successively m times. We achieve this through integration with respect to z and z', in which we substitute instead of ζ and ζ' the integration variables z and z' and introduce under the integral sign the factors $(\zeta - z)^{-m-1}$ and $(\zeta' - z')^{-m-1}$. With increasing m, we let the points ξ, η, ζ and ξ', η', ζ' approach the North pole *simultaneously*; more precisely said, we make the substitution

$$(3)\qquad \begin{aligned} \xi &= \frac{kx}{m}, \quad \eta = \frac{ky}{m}, \quad \zeta = 1 - \frac{k^2(x^2+y^2)}{2m^2} + \dots, \\ \xi' &= \frac{-kx'}{m}, \quad \eta' = \frac{-ky'}{m}, \quad \zeta' = 1 - \frac{k^2(x'^2+y'^2)}{2m^2} + \dots \end{aligned}$$

and understand that x, y, x', y' are finite and that m is a number that increases to infinity. In addition to the rectangular coordinates $x, \dots$, we need also the polar coordinates $r, \dots$, which are defined by

$$(4)\qquad \begin{aligned} r^2 &= x^2 + y^2, \quad e^{i\varphi} = \frac{x+iy}{r}, \\ r'^2 &= x'^2 + y'^2, \quad e^{i\varphi'} = \frac{x'+iy'}{r'}. \end{aligned}$$

For the purpose of the limit process we write $\frac{ikr\cos\alpha}{m}$ and $\frac{-ikr'\cos\alpha'}{m}$ instead of the variables z and z', respectively. Then there follows, in analogy to (12) on page 334, the double integral

$$(5)\qquad \frac{1}{2\pi}\int d\alpha\, e^{ikr\cos\alpha}\,\frac{1}{2i}\int d\alpha'\, e^{-ikr'\cos\alpha'}\,\frac{1}{1-e^{i(\varphi-\varphi'+\alpha+\alpha')}}.$$

The first integral should be taken over the path (A) and the second over the path (B) (the latter, however, with a modification given below; cf. the footnote on page 350). We have also introduced the numerical factors with which these integrals should be provided according to the convention in §4.

By stereographic projection from the North pole, there results in the same manner the double integral[32]

$$(6)\qquad -\frac{1}{2\pi}\int d\alpha\, e^{ikr\cos\alpha}\,\frac{1}{2i}\int d\alpha'\, e^{-ikr'\cos\alpha'}\,\frac{1}{1-e^{i(\varphi-\varphi'-\alpha+\alpha')}}.$$

The sum of the two we write in the form

$$U = \frac{1}{2\pi i}\int d\alpha\, e^{ikr\cos\alpha}\,\frac{1}{2i}\int d\alpha'\, e^{-ikr'\cos\alpha'}\,\frac{\sin\alpha}{\cos\alpha - \cos(\varphi-\varphi'+\alpha')}.$$

If we consider the integrand as a function of $\cos\alpha$, then we can evaluate the integral over α; namely, the path (A) can be contracted in the $\cos\alpha$ plane to a point $\cos\alpha = \cos(\varphi - \varphi' + \alpha')$, as discussed in §5 section 1. According to Cauchy's theorem, we thus obtain

$$(7)\qquad U = \frac{1}{2i}\int d\alpha'\, e^{ikr\cos(\varphi-\varphi'+\alpha')-ikr'\cos\alpha'}.$$

This function depends only on the straight line distance

$$R = \sqrt{(x-x')^2 + (y-y')^2}$$

between the points x, y and x', y'; namely, we can bring it into the form*)

$$U = \frac{1}{2i} \int d\gamma \, e^{ikR\cos\gamma} = U_0(kR), \tag{8}$$

from which we see that our function U becomes directly equal to the Bessel function of the 2nd kind with index 0 and argument kR.

We deduce immediately from equation (8) the following properties of our function.

a) *It satisfies the differential equation* $\Delta U + k^2 U = 0$ *due to its derivation from the complex function* f.

b) *It is single-valued in the smooth plane.*

c) *It is finite and continuous for all finite values of* x *and* y, *except at the pole* $x = x'$, $y = y'$, *where it becomes logarithmically infinite* (cf. equation (8), page 340). Therefore we have indeed been able to keep the singularity in the finite region through the modification of our general procedure given in §3 section 2.

d) *It vanishes at infinity.* The latter follows from the known semiconvergent expansion**) of the Bessel function for large values of the argument, or here even more simply from the fact that the path (B) can be taken completely in the hatched region where $\lim_{R=\infty} e^{ikR\cos\gamma}$ vanishes.

e) *It does not change when we interchange the coordinates of the variable point with the coordinates of the pole.* This property follows from the theorem of the interchange of parameter and argument in the theory of Abel functions***).

Our function $U_0(kR)$ is therefore always to be applied in optics when one considers the *exact description of a two-dimensional light motion that is caused by a luminous*

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*) As a proof one first writes β instead of $\varphi' - \alpha'$; one then expresses according to (4) the polar coordinates in terms of the rectangular coordinates. Furthermore, one sets $x - x' = R\cos\psi$, $y - y' = R\sin\psi$, so that ψ denotes the observation angle, with respect to the x-axis, of the point x, y as it appears from x', y'. Finally, one writes γ instead of $\beta - \psi$. Then (8) results from (7).

A more precise determination of the integration path in (7) must now be made in such a way that the path in (8) is directly equal to the path (B). The path in (7) is thus determined according to the equation $-\alpha' = \gamma + \psi - \varphi'$. In (7) we can therefore use the path (B) directly only if the point x, y lies on the half-ray $\psi = \varphi'$ or in its neighborhood. In the case of variability of the point x, y, the integration path changes continuously with the location of the point.

**) Cf. H. Hankel: Cylinderfunctionen erster and zweiter Art. Math. Ann. Bd. I, §6.

***) Cf. F. Pockels: l. c. p. 283, where the corresponding property for "Green's functions" of arbitrarily bounded regions is stated.

point lying at the finite location (x', y'). However, one does not use the function U_0 itself in ordinary optics, but rather one is unknowingly satisfied with the first term of its semiconvergent expansion, which in itself is not at all a solution of our differential equation.

2. In order to now proceed to the construction of the *analogous function for our Riemann surface with an* $(n-1)$*-fold winding point*, we will begin not with the function (1) that is single-valued in the smooth plane, but rather, as in §5 section 2, with the function

$$f(\mathsf{Z}) = \frac{\frac{1}{n}}{1 - \left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{1}{n}}}$$

that results from (1) if we map our Riemann surface with respect to the variables Z and Z' conformally to the Z and Z' planes, respectively. We treat this function exactly as the function (1) in the preceding. We immediately give the result in the definitive form[33]

$$U = \frac{1}{2i\pi n} \int \frac{\sin\frac{\alpha}{n}}{\cos\frac{\alpha}{n} - \cos\frac{\varphi - \varphi'}{n}} U_0(kR')\, d\alpha \tag{9}$$

$$[R'^2 = r^2 + r'^2 - 2rr'\cos\alpha],$$

where we have brought the originally appearing double integral into the form of a simple integral through the introduction of the Bessel function U_0. The integration path is the path (A).

We must first convince ourselves *that this integral has, in general, a finite value.* Since the function U_0 appears here with a complex argument, it is important to know its behavior in the complex plane. Concerning this, we note: 1) the function $U_0(z)$ is *branched* at the point $z = 0$ in the complex z plane; namely, it is increased by $2\pi i J_0(z)$ when that point is encircled;[34] 2) it behaves at infinity as $-\sqrt{\frac{\pi}{2z}}\, e^{i(z+\frac{\pi}{4})}$, *and therefore vanishes at infinity in the positive half plane* (that is, that half plane where z has a positive imaginary part).[35] The former follows from equation (6), page 339, and the latter from the semiconvergent series for Bessel functions*).

*) H. Hankel: l. c.

To the branch point $R' = 0$ there corresponds in the α plane an infinite number of point pairs

$$\alpha = \pm ia + 2k\pi \quad \left[k = -\infty \text{ to } +\infty, a = \arccos \frac{r^2 + r'^2}{2rr'}\right],$$

of which the pair $\alpha = \pm ia$ lying on the imaginary axis is of primary interest for us. In the same way there are naturally infinitely many realizations of the

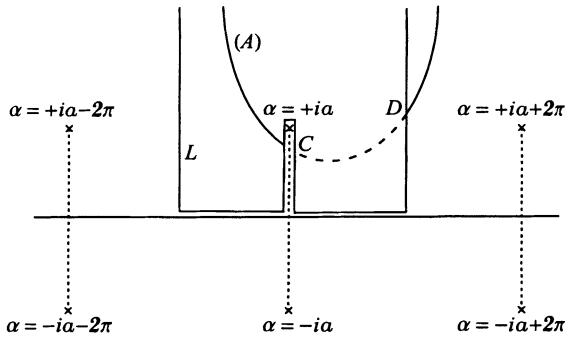

Fig. 4.

real axis[36] in the α plane. We have illustrated one of these. It is the segmented line L $(-\pi + i\infty, -\pi, 0, ia, 0, \pi, \pi + i\infty)$. The positive R' half plane corresponds to the interior of the region enclosed by L. When crossing L, one comes each time to a part corresponding to the negative R' half plane.

We now wish to make our above specification of the integration path more precise, in that it leaves the point $\alpha = ia$ to its left and the singular points lying on the real axis to its right. Otherwise it can obviously be deformed arbitrarily, except that it may not be pulled over the points $\alpha = \pm ia, \pm(\varphi - \varphi') + 2kn\pi$.

The dashed part (CD) of the integration path in Fig. 4 corresponds in the R' plane to points of the negative half plane. The beginning and end of the integration path are drawn in full lines to indicate that the corresponding points in the R' plane lie in the positive half plane. The function $U_0(kR')$ therefore vanishes in the infinitely distant parts of the integration path.

From this we see that the integral (9) generally has a finite value; an exception occurs only in the special case $r = r'$, $\varphi = \varphi'$. Namely, if we let $r = r'$, then the two branch points $\alpha = \pm ia$ come together into the one point $\alpha = 0$. The integration path, which is so to speak pinned between the two points, then necessarily passes through the

point $\alpha = 0$. Because of this the integral does not necessarily have to become infinite. However, if we also let $\varphi = \varphi'$, then the factor $\dfrac{\sin\frac{\alpha}{n}}{\cos\frac{\alpha}{n} - \cos\frac{\varphi-\varphi'}{n}}$ becomes infinite at this point. *Then and only then does the integral become infinite.*

To investigate the type of singularity, we construct the symmetric function $U_1 + U_2 + \cdots + U_n$ of the values of U at the overlying points of the surface. Here we designate, as previously, the collection of those points for which $|\varphi - \varphi'| < \pi$ as the first sheet, and as the $2^{\text{nd}} \ldots n^{\text{th}}$ sheet all remaining points. U_1 corresponds to the first sheet, and $U_2 \ldots U_n$ to the $2^{\text{nd}} \ldots n^{\text{th}}$ sheet. According to the calculation given in §5 on page 344, there now follows

$$U_1 + U_2 + \cdots + U_n = \frac{1}{2\pi i}\int \frac{\sin\alpha}{\cos\alpha - \cos(\varphi-\varphi')} U_0(kR')\, d\alpha.$$

If we conceive of this expression as an integral in the complex plane of the variable $\cos\alpha$, then we can evaluate it by Cauchy's theorem. Namely, the integral is a single-valued function of $\cos\alpha$; we obtain

$$U_1 + U_2 + \cdots + U_n = U_0(kR) \quad [R^2 = r^2 + r'^2 - 2rr'\cos(\varphi - \varphi')].$$

From equation (8) on page 340, the function $U_0(kr)$ becomes logarithmically infinite for $R = 0$. According to the preceding, however, the quantities $U_2 \ldots U_n$ certainly have finite values. The singularity must therefore be attributed only to U_1. Thus the function U becomes infinite as $\lg\frac{1}{R}$ at the point $r = r'$, $\varphi = \varphi'$ on the first sheet.

Accordingly, we can state the following about our function U:

a) *It satisfies the differential equation* $\Delta U + k^2 U = 0$.

b) *It is single-valued on our n-sheeted winding surface.*

c) *It is finite and continuous for all finite values of* x *and* y *except at the point* $x = x'$, $y = y'$, *where it has a simple pole.*

d) *It vanishes at infinity.* Namely, if $r = \infty$, then the points $\alpha = \pm ia$ move in the same way along the imaginary axis infinitely far away from each other. We may then move the integration path arbitrarily far in the direction of the positive imaginary axis, whereby the integral becomes arbitrarily small due to the factor $\dfrac{1}{\cos\frac{\alpha}{n} - \cos\frac{\varphi-\varphi'}{n}}$.

e) *The function* U *remains unchanged if we interchange the roles of the variable point and the pole.*

f) *The symmetric function $U_1 + U_2 + \cdots + U_n$ of the values of U at the overlying points of the surface is equal to the analogous single-valued solution $U_0(kR)$.*

We remark further that there is complete agreement in the manner of constructing the functions u, u_0 in the previous paragraph and the functions U, U_0 in this paragraph. The function u, we can say, resulted from the single-valued solution u_0 when we replaced the real quantity $\cos(\varphi - \varphi')$ by the complex $\cos\alpha$, introduced the factor $\dfrac{\frac{1}{n}\sin\frac{\alpha}{n}}{\cos\frac{\alpha}{n} - \cos\frac{\varphi-\varphi'}{n}}$, and integrated with respect to α over the path (A). The function U results from the single-valued function U_0 in exactly this way.

The function U represents *a light motion on our Riemann surface that is caused by a luminous point at the position r', φ'*. It must take the place of the function U in those optical problems that by their nature give rise to functions that are single valued only through their analytic continuation on our Riemann surface; namely, in the problem of diffraction by a wedge-shaped opaque body in the presence of a luminous point lying in the finite region.[37]

3. According to §3, all solutions of our differential equation that are regular in the interior of a circle or a circular ring may be expanded in a series that progresses in Bessel functions of the 1st or 2nd kind, respectively. We now wish to actually construct these expansions for our function U. (The function U_0 need not be considered separately, since it represents only a special case of U.) And indeed, *we will derive them directly from the power series expansion of the complex starting function.* However, we do not obtain a single expansion, as for the function u of the preceding paragraph, but rather we must distinguish *two regions* in which different expansions hold; namely, the *interior and exterior of the circle $r = r'$*.

The function U is calculated as the sum of two integrals, of which one is obtained by projection of the function f from the South pole, and the other is obtained by projection from the North pole (in the same way as is explicitly carried out for the function U_0 above; cf. (5) and (6) on page 349). Correspondingly, we consider, with the use of the notation from §5 section 3,

$$f(\mathsf{Z}) + f(\mathsf{Z}_1) = \frac{1}{n}\left\{\frac{1}{1-\left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{1}{n}}} + \frac{1}{1-\left(\frac{\mathsf{Z}_1}{\mathsf{Z}'}\right)^{\frac{1}{n}}}\right\}. \tag{10}$$

Here there exist between the radius vector r of a point of the x, y plane and the radius vectors $|\mathsf{Z}|$, $|\mathsf{Z}_1|$ of those points of the complex plane, which are transformed into the point x, y by one or the other of the projections, the relations, as on page 345,

(11a) $$|\mathsf{Z}| = \frac{kr}{m}, \quad |\mathsf{Z}_1| = \frac{m}{2kr}.$$

The corresponding equation holds between r' and $|\mathsf{Z}'|$, namely

(11b) $$|\mathsf{Z}'| = \frac{kr'}{m}.$$

1. $r < r'$. From equations (11a) and (11b) there follow

$$|\mathsf{Z}| < |\mathsf{Z}'| \quad \text{and} \quad |\mathsf{Z}_1| > |\mathsf{Z}'|.$$

In this case we have to assume for the function (10) the convergent power series expansion

$$f(\mathsf{Z})+f(\mathsf{Z}_1) = \frac{1}{n}\left\{1+\left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{1}{n}}+\left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{2}{n}}+\cdots-\left(\frac{\mathsf{Z}'}{\mathsf{Z}_1}\right)^{\frac{1}{n}}\left(1+\left(\frac{\mathsf{Z}'}{\mathsf{Z}_1}\right)^{\frac{1}{n}}+\cdots\right)\right\}.$$

We integrate the left and right sides of this equation on the path (A) with respect to α and on the path (B) with respect to α'*).[38] Then there follows on the left our function U. On the right, we perform the integration term by term. We use the summary at the end of §4; there follow

from $\left(\frac{\mathsf{Z}}{\mathsf{Z}'}\right)^{\frac{m}{n}}$	$e^{\frac{im}{n}(\varphi-\varphi')} J_{\frac{m}{n}}(kr)U_{\frac{m}{n}}(kr')$,
from $\left(\frac{\mathsf{Z}'}{\mathsf{Z}_1}\right)^{\frac{m}{n}}$	$-e^{-\frac{im}{n}(\varphi-\varphi')} J_{\frac{m}{n}}(kr)U_{\frac{m}{n}}(kr')$,
from 1	$J_0(kr)U_0(kr')$.

Thus we obtain

(12) $$U = \sum_{m=0}^{m=\infty} \gamma_m J_{\frac{m}{n}}(kr)U_{\frac{m}{n}}(kr') \cos\frac{m}{n}(\varphi-\varphi')$$

$$\left[\begin{array}{l} \gamma_m = \frac{2}{n} \text{ for } m > 0 \\ \gamma_m = \frac{1}{n} \text{ for } m = 0 \end{array}\right].$$

We thus arrive at *an expansion valid for the interior of the circle $r = r'$ that progresses in Bessel functions of the first kind*, in agreement with the general

*) For the meaning of this abbreviated manner of expression, cf. the footnote on pag. 346.

theorem of §2, page 325. The series appears there only in a somewhat different form, in that $J_{\frac{m}{n}}(kr)$ is multiplied there not by $U_{\frac{m}{n}}(kr')$ but rather by $1 : J_{\frac{m}{n}}(kr')$. However, if one compares the given limiting values of the Bessel functions of the first and second kind as the index increases to infinity, then one sees that $\lim_{\nu=\infty} U_\nu : \frac{1}{J_\nu}$ is smaller than $\frac{1}{\nu}$ multiplied by a number that is independent of ν. Therefore, our series (12) always converges when the series (2) of §2 converges; that is, in the interior of the circle of radius r', and truly represents the function U here.

2. $r > r'$. If we consider for the moment not r, φ but rather r', φ' as the variable point, then the series (12) yields at the same time a representation of $U(r', \varphi')$ through Bessel functions of the second kind that is valid, as a consequence of the derivation, as long as $r' > r$. Now the function U remains unchanged when r', φ' are interchanged with r, φ, on the basis of its reciprocity property e). We arrive then at the series

$$(13) \qquad U(r,\varphi) = \sum_{m=0}^{m=\infty} \gamma_m J_{\frac{m}{n}}(kr') U_{\frac{m}{n}}(kr) \cos \frac{m}{n}(\varphi - \varphi')$$

$$\left[\begin{array}{l} \gamma_m = \frac{2}{n} \text{ for } m > 0 \\ \gamma_m = \frac{1}{n} \text{ for } m = 0 \end{array} \right]$$

which is valid for all finite points outside the circle $r = r'$. This is a series of the form considered in §2 section 5, only here $J_{\frac{m}{n}}(kr')$ has taken the place of $1 : U_{\frac{m}{n}}(kr')$. According to our remark above on the limiting value of $J_\nu U_\nu$ as ν increases to infinity, this only improves the convergence of the series. Thus the series (13) converges for all values $\infty > r > r'$. That such a series actually represents the given solution in this region could not generally be asserted. But it follows here because of the particular reciprocity property of U and the fact that the series (12) actually represents the function U in the region $r < r'$.

If we separate the series (12) and (13) into their real and imaginary parts according to equation (7) on page 340, then there result in the simplest case $n = 1$, that is in the case of the single-valued solution $U_0(kR)$, the so-called *addition theorems for the Bessel functions* $J_0(kR)$ *and* $K_0(kR)$, as they are presented by Heine, though without establishing their regions of validity. We remark further that

the corresponding *addition theorems for spherical functions* also result in the same way, if we interrupt our procedure before the execution of the limit process.

4. It hardly needs to be mentioned that our starting assumption also yields the function with a simple pole that is single-valued *on an infinitely-sheeted winding surface* in the form

$$U = \frac{1}{2\pi} \int \frac{2\alpha \, d\alpha}{(\varphi - \varphi')^2 - \alpha^2} U_0(kR').$$

For this function the properties a) – f) on pages 353 and 354 are equally valid mutatis mutandis, and the previously considered solutions U can be constructed from it in reverse.

§7.
Graphical treatment of the simplest multivalued solution.

This paragraph pursues the two-fold purpose of attempting to give a clear picture of the form of our multivalued solutions in a special case and preparing the formulas for numerical calculation.

For numerical calculation, neither the representation by a complex integral nor the series of Bessel functions is suitable. As we have seen, the latter certainly converge for all finite points of the x, y plane in the case of the everywhere finite solutions. However, the convergence becomes poorer with increasing values of the argument kr. Now $\frac{k}{2\pi}$ means physically, as remarked on page 318, the reciprocal wavelength of the considered oscillation. The convergence will therefore always be poor when the distance r from the winding point amounts to a large number of wavelengths.

For optical applications, where the wavelength is so extraordinarily small, our *convergent* series are certainly not useful. For acoustic or Hertzian oscillations, they may perhaps be of use. Nevertheless, we will immediately give *divergent* series that are very practical for optical purposes. In doing so, we limit ourselves to the simplest and at the same time most important case of the solution u that is single-valued on a two-sheeted winding surface and finite everywhere.

We first transform the complex integral (5), through which this solution was given on page 343, into an integral on a real path. For brevity, we wish to set $\varphi - \varphi' = \psi$, or, in other words, we wish to choose the "incident direction of

the wave" as the x-axis, so that r and ψ now denote the usual polar coordinates. We designate, as earlier, the points for which $|\psi| < \pi$ or $|\psi| > \pi$ as the first or second sheet, respectively. According to e) on page 344, we have

$$u_1 + u_2 = u_0 = e^{ikr\cos\psi}. \tag{1}$$

Further, one calculates from (5) on page 343

$$u_1 - u_2 = \frac{\cos\frac{\psi}{2}}{\pi i}\int \frac{e^{ikr\cos\alpha}\sin\frac{\alpha}{2}\,d\alpha}{\cos\alpha - \cos\psi}, \tag{2}$$

where $|\psi| < \pi$ is supposed. If we temporarily let

$$X = \frac{u_1 - u_2}{u_0},$$

there follows

$$\begin{aligned}\frac{\partial X}{\partial r} &= \frac{k}{\pi}\cos\frac{\psi}{2}\int e^{ikr(\cos\alpha-\cos\psi)}\sin\frac{\alpha}{2}\,d\alpha \\ &= \frac{k}{\pi}\cos\frac{\psi}{2}\; e^{-2ikr\cos^2\frac{\psi}{2}}\int e^{2ikr\cos^2\frac{\alpha}{2}}\sin\frac{\alpha}{2}\,d\alpha.\end{aligned}$$

If we consider $\cos\frac{\alpha}{2}$ as the integration variable in the last integral and deform the integration path, as it results for this variable from the path (A) in the α plane,[39] into the imaginary axis of the $\cos\frac{\alpha}{2}$ plane, then we can evaluate the integral; it becomes

$$\int e^{2ikr\cos^2\frac{\alpha}{2}}\sin\frac{\alpha}{2}\,d\alpha = \sqrt{\frac{2\pi}{kr}}\,e^{\frac{i\pi}{4}}.$$

Therefore

$$\frac{\partial X}{\partial r} = \sqrt{\frac{2k}{-i\pi r}}\cos\frac{\psi}{2}\,e^{-2ikr\cos^2\frac{\psi}{2}} = \frac{\partial}{\partial r}\frac{2}{\sqrt{-i\pi}}\int_0^{\sqrt{2kr}\cos\frac{\psi}{2}} e^{-i\tau^2}\,d\tau.$$

If we integrate between 0 and r, there follow, since $X = 0$ for $r = 0$,

$$X = \frac{2}{\sqrt{-i\pi}}\int_0^T e^{-i\tau^2}\,d\tau$$

and

$$u_1 - u_2 = \frac{2}{\sqrt{-i\pi}}\,e^{ikr\cos\psi}\int_0^T e^{-i\tau^2}\,d\tau, \quad T = \sqrt{2kr}\cos\frac{\psi}{2}, \tag{3}$$

where the integral is now taken on a real path. We can also write equation (1) as

$$u_1 + u_2 = \frac{2}{\sqrt{-i\pi}}\,e^{ikr\cos\psi}\int_{-\infty}^0 e^{-i\tau^2}\,d\tau. \tag{4}$$

From equations (3) and (4) there follow expressions for u_1 and u_2 that hold in the first place for $|\psi| < \pi$, but which are transformed into each other if we exchange ψ with $\psi + 2\pi$. We thus obtain a single expression for u that is valid in the first as well as the second sheet if we remove the requirement $|\psi| < \pi$; namely,

$$u = e^{ikr\cos\psi}\,\frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\int_{-\infty}^{T} e^{-i\tau^2}\,d\tau, \quad T = \sqrt{2kr}\cos\frac{\psi}{2}. \tag{5}$$

The quantity T is positive on the first sheet and negative on the second sheet.

On the basis of this formula, it is now possible to obtain an intuitive picture of the form of the function u and to sketch a drawing. Since intuition and drawings can capture the properties of a function only imprecisely, we will allow ourselves here to be satisfied with approximate values, but in such a way that we can estimate the committed error precisely. *Accordingly, for the entire remainder of this work, a specified error ε shall be decisive.* The magnitude of ε can be prescribed arbitrarily. All the following propositions are correct only within an error that is smaller in absolute value than ε. Thus, an abbreviated manner of expression will be allowed. We will say that u is equal to a given value when we must say more precisely that the difference between the given and the true value of u is smaller in absolute value than ε.

Approximate calculation of u on the second sheet. It is first necessary to find approximate values for the integral that appears in (5). We develop these through n-fold partial integration in the series

$$\frac{1}{\sqrt{\pi}}\int_{-\infty}^{-|T|} e^{-i\tau^2}\,d\tau = \frac{1}{\sqrt{\pi}}\int_{|T|}^{\infty} e^{-i\tau^2}\,d\tau \tag{6}$$

$$= \frac{e^{-iT^2}}{\sqrt{\pi}}\left\{\frac{1}{2i}\frac{1}{|T|} - \frac{1}{(2i)^2}\frac{1}{|T|^3} + \frac{1\cdot 3}{(2i)^3}\cdot\frac{1}{|T|^5} - \cdots\right.$$

$$\left. + (-1)^{n-1}\frac{1\cdot 3\ldots 2n-3}{(2i)^n}\frac{1}{|T|^{2n-1}} + (-1)^n\frac{1\cdot 3\ldots 2n-1}{(2i)^n}\int_{|T|}^{\infty} e^{-i\tau^2}\,\frac{d\tau}{\tau^{2n}}\right\},$$

where we have set $T = -|T|$, since T is negative on the second sheet.

If one lets n increase to infinity, then the series becomes *divergent*. Thus one may go only to a finite n and must consider the final remainder term. This is always smaller than the preceding term, as is shown by a simple observation, so that the series (6) may be designated as *semiconvergent*.

We will use for an approximate calculation merely the *first* term. This is allowed for the case in which the second term, taken in absolute value, is already less than $\frac{\varepsilon}{2}$; that is, when

$$\left| e^{-iT^2} \frac{1}{\sqrt{\pi}(2i)^2|T|^3} \right| = \frac{1}{4\sqrt{\pi}|T|^3} < \frac{\varepsilon}{2} \quad \text{or} \quad \frac{1}{|T|^3} < 2\sqrt{\pi}\varepsilon.$$

For then the entire remainder is also less than $\frac{\varepsilon}{2}$ and the committed error is less than ε. It is convenient to replace this condition by a somewhat stronger one, namely by

$$\frac{1}{4T^2} < \varepsilon. \tag{7}$$

The lower sheet is divided for our further consideration *into two regions, depending on whether this inequality is fulfilled or not.* We call that region in which the inequality is not fulfilled the *"transition strip."* The boundary of the transition strip is given by the equation

$$\frac{1}{8kr\cos^2\frac{\psi}{2}} = \varepsilon \quad \text{or} \quad r(\cos\psi + 1) = \frac{\lambda}{8\pi\varepsilon},$$

where $\lambda = \frac{2\pi}{k}$ denotes the wavelength. The boundary curve is a *parabola.* In rectangular coordinates $x = r\cos\psi$, $y = r\sin\psi$ this equation is namely

$$y^2 + 4e(x - e) = 0, \quad e = \frac{\lambda}{16\pi\varepsilon}, \tag{8}$$

where e denotes the distance from the vertex to the focal point. The vertex lies at the point $x = e$, $y = 0$, and the focal point $x = 0$, $y = 0$ coincides with the winding point. The parabola encloses the negative part of the x-axis. In order to have specific values in mind, we wish to assume for λ the approximate value of the D-line $5 \cdot 10^{-4}$ mm,[40] for 16π the number 50, and to determine the precision in such a way that $\varepsilon = \frac{1}{1000}$. Then $e = \frac{1}{10}$ mm. The parabola is thus extremely thin*).

The inequality (7) is satisfied at all points outside the parabola. Thus it is sufficient, under the conditions that exist in optics,

*) If the wavelength or the prescribed precision becomes larger (that is, if λ is larger or ε is smaller), the parabola assumes somewhat larger dimensions. Thus it occurs that the transition region appears rather large in the figures of the plate. The wavelength here had to be taken to be at least 10 mm for the figure to be clear. The precision was chosen as $\varepsilon = 2 \cdot 10^{-2}$, so that e was determined as $e = 10$ mm.

to replace the series (6) by its first term for the greater part of the second sheet.

For u itself there results correspondingly from (5) the approximate value

$$u = \sqrt{\frac{\lambda}{r}}\,\frac{e^{-\frac{2\pi i r}{\lambda}+\frac{\pi i}{4}}}{4\pi i\left|\cos\frac{\psi}{2}\right|}$$

or

$$u = -\sqrt{\frac{\lambda}{r}}\,\frac{e^{-\frac{2\pi i r}{\lambda}-\frac{i\pi}{4}}}{4\pi\cos\frac{\psi}{2}}. \tag{9}$$

This equation is valid with precision ε at all points in the second sheet that do not belong to the transition strip.

Approximate computation of u on the first sheet. On the first sheet, where $T > 0$, we cannot use the series (6) directly. But according to equation (1) on page 358, we have

$$u_1 = u_0 - u_2.$$

We also delimit a *transition region* on the first sheet, in that we carry over our parabola into the corresponding location on the first sheet. For all points of the exterior of the parabola, we can then replace u_2 by the approximate value from equation (9). Thus we obtain for u the approximate representation

$$u = e^{\frac{2\pi i r}{\lambda}\cos\psi} - \sqrt{\frac{\lambda}{r}}\,\frac{e^{-\frac{2\pi i r}{\lambda}-\frac{i\pi}{4}}}{4\pi\cos\frac{\psi}{2}}, \tag{10}$$

which is valid at all points of the first sheet that do not belong to the transition region.

In the transition regions of the first and second sheets, on the contrary, we must use the exact formula (5).[41]

In the following, we wish to make use of the terminology of optics. Now the function u does not yet have a physical meaning in itself; it corresponds to the auxiliary functions $\Xi, \mathsf{H}, \mathsf{Z}$ introduced in §1. But we obtain a quantity that we can designate as the light oscillation, and that corresponds to the quantities X, Y, Z in §1, if we form

$$\mathfrak{Re}\left(e^{\frac{2\pi i t}{\tau}}u\right), \tag{11}$$

where t denotes time and τ is the period of the oscillation. We will arrive at a picture of the resulting oscillatory state if we construct its *level curves* at

various points of time. In the simplest case this is achieved for the *nodal lines*: that is, for those curves in which the oscillation is 0. *We ask therefore for the form of these null lines in the first and second sheets, and first of all for the time* $t = 0$. These are represented by the equation $\mathfrak{Re}(u) = 0$.

Outside the transition region in the *second sheet* the null lines have a very simple form. According to (9), their equation here is

$$(12)\quad \sqrt{\frac{\lambda}{r}}\,\frac{1}{4\pi\cos\frac{\psi}{2}}\cos\left(\frac{2\pi r}{\lambda}+\frac{\pi}{4}\right)=0;\quad \text{that is,}\quad \cos\left(\frac{2\pi r}{\lambda}+\frac{\pi}{4}\right)=0$$

$$\text{or}\quad r=\lambda\left(\frac{m}{2}+\frac{1}{8}\right)\quad [m=1,2,3,\ldots].$$

They form a system of concentric circles around the winding point in which the radii of successive circles differ by half a wavelength. We draw these circles in Fig. 1 of the plate,[42] where we can extend each circle only to the transition region. For what concerns the path of the null curves in the transition region, we can say nothing according to (9). (It hardly needs to remarked that in truth the null curves are not exact circles, but rather are transcendental curves. Precisely, we can only say that along each of these circles, the value of $|u|$ is smaller than ε.)

The null curves in the *first sheet* outside the transition region are calculated from equation (10). Their equation for $t = 0$ is, if we set $r\cos\psi = x$ in the first term,

$$(13)\qquad \cos\frac{2\pi x}{\lambda}-\sqrt{\frac{\lambda}{r}}\,\frac{1}{4\pi\cos\frac{\psi}{2}}\cos\left(\frac{2\pi r}{\lambda}+\frac{\pi}{4}\right)=0.$$

The first term (u_0), itself set equal to 0, yields a *system of parallel lines* G that are perpendicular to the x-axis and follow one another at a distance $\frac{\lambda}{2}$. These are simply the nodal lines of an oscillation that would occur if the winding point were not present. We draw these lines in Fig. 2 of the plate. The second term (u_2), itself set equal to 0, yields the depicted system of concentric circles. We now have to make clear to ourselves the null curves that correspond to the combined expression $u_0 - u_2 = 0$. For this purpose, we imagine Fig. 1 superposed on the system of lines in Fig. 2, and first consider one of those points P in which a circle and a line intersect. According to equation (13), the newly generated null curve must also pass through this point.

Furthermore, the null curve between two successive points P at a large distance from the winding point can deviate only slightly from the

lines G, because u_2 is small in relation to u_0 due to the factor $r^{-\frac{1}{2}}$. The new null line is consequently pulled only slightly toward that side where u_0 and u_2 have the same sign, and will in general undulate around the line G, passing sequentially through all points P that lie on the line G.

It could be possible, however, that the null lines in greater proximity to the winding point are deflected from their corresponding lines G so much that they join two points P that originally lie on different lines G. In fact, this is not the case. Namely, let us consider one of those lines G' that runs in the middle between two successive lines G. On this line u_0 has the value $+1$ or -1. On the other hand, $|u_2|$ has the value $\frac{\vartheta}{T}$, where $\vartheta = \frac{1}{\sqrt{4\pi}} \cos\left(\frac{2\pi r}{\lambda} + \frac{\pi}{4}\right)$ is a proper fraction and $1 : T$, according to (7), is smaller than $2\sqrt{\varepsilon}$ and therefore also a proper fraction, in so far as we prescribe a reasonable precision. Thus $|u_2| < 1$, and $u_0 - u_2$ cannot vanish anywhere along the lines G'. Therefore the resulting null curve cannot cross the line G' anywhere, even near the winding point. *The null curves follow, in essence, the course of the lines G, in that they cross all points P successively, and oscillate between these points with deviations smaller than* $\frac{\lambda}{4}$, *becoming more and more flat as the null curve moves farther from the winding point.* Accordingly, we cover the top sheet with such a system of serpentine curves in Fig. 2, where we may trace each null curve only until the transition strip.

Now for what concerns the *transition region*, we must revert to the exact formula (5). Here we can easily first find *the intersection points of the null lines with the axis of our parabola.* Namely, on the axis of the parabola $\psi = \pm\pi$, and therefore $T = 0$. The intersection points are therefore given by the equation

$$\mathfrak{Re}\,(u) = \cos\frac{2\pi r}{\lambda} = 0 \quad \text{or} \quad r = \frac{\lambda}{4} + \frac{m\lambda}{2} \quad (m = 0, 1, 2, \ldots).$$

These null points (Q_m) follow each other at the constant distance $\frac{\lambda}{2}$, and in particular one finds the same points Q_m in the first and second sheets. At each of these points, the corresponding null curve has a *corner.* It is not difficult to determine the magnitude of this corner from equation (5). The formula gives $\pi\sqrt{\frac{2m+1}{2}}$ for the tangent of the half angle of the corner in the first sheet and $-\pi\sqrt{\frac{2m+1}{2}}$ in the second sheet. The corners

are therefore opened in opposite directions in the first and second sheet; their angle becomes more and more flat with increasing distance from the winding point.

Now the branches of the null curve that clash onto each other in Q_m must pass into one of the previously described null curves outside the transition region. We place one such connection between the corners Q_m and the previously drawn pieces of the successive null curves in Figs. 1 and 2 in an arbitrary way, without concerning ourselves with the specific form of the connecting curves. The drawing of the system of the level curves is thus complete.[43]

By comparison of Figs. 1 and 2, one can see that the null curves of one sheet pass continuously with a continuous tangent onto those of the other sheet. The previously mentioned corners are therefore only apparent, and have appeared only because the lines $\psi = \pi$ and $\psi = -\pi$, which are different on the Riemann surface, coincide in the figures.

Let us now describe the *complete path of a null curve in the first and second sheets*, where we wish to choose one of those curves that intersects the axis of the parabola. We start at infinity of the first sheet above the axis of the parabola. The null curve first clings infinitely close to the line G, and then oscillates around this line with increasing amplitude as it approaches the winding point. Upon entering the transition strip, it is pulled toward the winding point. After it has intersected the axis of the parabola at one of the points Q_m, it passes onto the second sheet, and loops around the winding point in a circle. In its second entrance into the transition region, it moves away somewhat from the winding point, and arrives back on the first sheet at the point Q_m, where with decreasing oscillations it essentially follows the direction of the line G and is lost at infinity below the axis of the parabola. Those null curves that do not intersect the axis of the parabola have a simpler path, in that they do not encircle the winding point.[44]

This is the state of the oscillation at time $t = 0$. If we now wish to also visualize *its dependence on t*, as given through formula (11), then we need not repeat the calculation. Rather, we need only imagine that the system of null curves drawn above wanders in the direction of decreasing x, and we can describe the changes that take place for a single null curve with increasing time in the following way. At the beginning of time our

null curve intersects the incident direction $\psi = 0$ at infinity of the first sheet. It appears here as a straight line G. In approaching the winding point, it develops periodically repeating deformations that are strongest in those portions close to the winding point. At the moment when it passes the winding point, the deformation degenerates into a cusp that then resolves into a loop. The part of the loop that lies in the second sheet has the form of a circle, while those two ends of the loop that reach into the first sheet conform more or less to a line G. In wandering further, the double point on the axis of the parabola moves away from the winding point, while the ends resemble more and more a line that lies perpendicular to the axis of the parabola and the curvature of the loop becomes larger and larger.

Everything that is said here about the form of the null curves is naturally valid only *within the accuracy of the initially prescribed precision.* However, the characteristic features given here remain valid in essence if one requires a greater precision. We must then only enlarge our transition region correspondingly and restrict our numerical predictions to a correspondingly smaller part of the Riemann surface. If an absolute accuracy is prescribed, however, then our descriptions are not valid for any finite part of the surface.

In conclusion, let us remark explicitly that the oscillatory state has an essentially different character in the first and second sheets; while we can compare the light motion in the first sheet approximately with a *plane* wave, the light motion in the second sheet is much more similar to a *spherical wave* that appears to emanate from the winding point. It is almost as if a small part of the oscillating energy of the first sheet trickles through the winding point into the second sheet and there, similar to a stone upon a surface of water, excites a system of circular wave rings.

All these things will acquire real significance in the next paragraph.

§8.
Application to diffraction.

In this paragraph we treat the following simplest diffraction problem: let there be in space an infinitely thin, completely opaque, rectilinear plane screen S, of which the edge forms the z-axis.

Parallel light*) falls in a plane perpendicular to the edge of the screen. The oscillatory state is to be determined at each point of space.

With Mr. Poincaré**) we decompose the state into two simpler ones through which the original state can be reconstructed, and we characterize these states in the terminology of the electromagnetic theory of light in the following way: in case (a) the *electric* and in case (b) the *magnetic oscillation* is everywhere parallel to the z-axis.[45] Because Maxwell's equations for a nonconductive medium (in our case, air) are formed in a completely symmetric way, we can regard the first equations of §1 as differential equations for the electric as well as the magnetic vector. Both cases result, by virtue of $X = 0$ and $Y = 0$, in $\frac{\partial Z}{\partial z} = 0$. The state is therefore independent of one coordinate (z) and becomes two-dimensional. The differential equations for X, Y, Z reduce, if we pass from the Z-component to the t-independent function Z, to

$$\frac{\partial^2 \mathsf{Z}}{\partial x^2} + \frac{\partial^2 \mathsf{Z}}{\partial y^2} + k^2 \mathsf{Z} = 0. \tag{1}$$

The *boundary conditions* along S are different, however, in the two cases. If we assume with Mr. Poincaré that the screen is a perfect conductor, we obtain along S

$$\begin{aligned} &\mathsf{Z} = 0 \quad \text{in case (a)}, \\ &\frac{\partial \mathsf{Z}}{\partial n} = 0 \quad \text{" " (b)}. \end{aligned} \tag{2}$$

In both cases we add the condition that Z should be finite everywhere.[46]

One can satisfy the preceding conditions very easily with the help of our multivalued solutions. We imagine that the x, y plane is cut along S and a second plane is connected to it alternatively. For the resulting two-sheeted Riemann surface, S forms a "natural branch cut"; the branch point is the intersection point of the edge of the screen with the x, y plane.

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*) This manner of expression requires a clarification. If we interpret, as usual, the words "parallel light" through the function u_0 on page 342, then we must strictly speaking say: *parallel light is impossible in the presence of the screen.* For in the parallel path of the level curves there appears a disturbance caused by S. However, we may assume for physical reasons that the perturbed function is finite everywhere just as u_0 is, and that it behaves at infinity similarly to u_0. Nothing more should be meant by the above expression "parallel light."

**) Poincaré: Sur la polarisation par diffraction, Acta Math. B. 16, §1.

We may distinguish the two sheets as the physical sheet and the auxiliary sheet. Let the azimuth φ be measured from the screen S, so that $0 < \varphi < 2\pi$ in the physical sheet and $-2\pi < \varphi < 0$ in the auxiliary sheet. We now call upon the everywhere finite two-valued solution u from §5, which we may write as $u(\varphi')$ in order to make the dependence on the incident direction $\varphi - \varphi'$ explicit, and form

$$\begin{aligned} \mathsf{Z} &= u(\varphi') - u(-\varphi') \quad \text{in case (a)}, \\ \mathsf{Z} &= u(\varphi') + u(-\varphi') \quad " \quad " \quad \text{(b)}. \end{aligned} \tag{3}$$

Then, as follows from symmetry considerations, $\mathsf{Z} = 0$ for $\varphi = 0$ and $\varphi = 2\pi$ in case (a), and $\dfrac{\partial \mathsf{Z}}{\partial \varphi} = 0$ for $\varphi = 0$ and $\varphi = 2\pi$ in case (b). Furthermore, Z satisfies the differential equation (1) and the condition that it be everywhere finite. We therefore have in equations (3) the solution of our physical problem. The *diffraction phenomena* caused by the screen S, we can say, are nothing more than *"interference phenomena" between the wave motions* $u(\varphi')$ *and* $u(-\varphi')$, of which one is incident in the direction $\varphi = \varphi'$ of the physical sheet, and the other is incident in the direction $\varphi = -\varphi'$ of the auxiliary sheet.

While in the preceding paragraph we examined a single such wave, we now have to imagine the state that results from the superposition of two such waves. For this purpose, we first mark the continuation of the two incidence directions; that is, the half-rays $\varphi = \varphi' + \pi$ and $\varphi = -\varphi' + \pi$. In optics one denotes the former as the *boundary line of the geometric shadow*; we wish we to say: of the geometric shadow *of the first kind.* Correspondingly, we name the latter the boundary line of the geometric shadow *of the second kind.*

These designations, however, have a direct meaning only in the extreme case of geometric optics, where the wavelength becomes equal to zero and our constant k thus becomes infinite. In this case our function u, as one sees immediately from equation (5) on page 359, becomes equal to zero on one side of the shadow boundary, and on the other side is equal to the single-valued solution u_0. Only this limiting case corresponds to the customary view of the nature of light, according to which it is dark behind an opaque body and the motion of light proceeds laterally as if the body were not present. For us, since we must assume the wavelength to be a finite quantity, although a very small one, the concept *"shadow"* becomes completely illusory, and the words *"shadow boundary"* mean for us only a line along which one approximation formula loses its validity and must be replaced by another.[47]

We separate furthermore two transition strips S_1 and S_2 by parabolas that have as their axes the shadow boundaries of the first and second kind, respectively, and which are given by equation (8) on page 360. The quantity ε denotes the precision prescribed for the calculation of the diffraction phenomena, as it might be given, for example, by the precision of the observations.*)

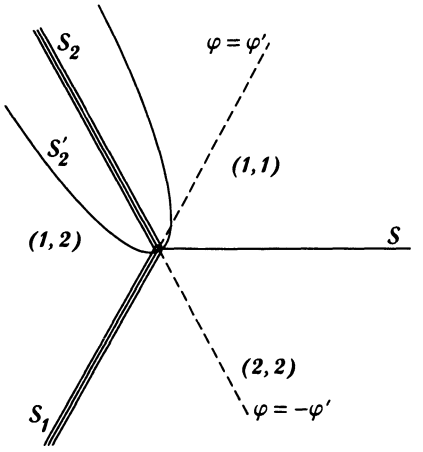


Fig. 5.

The physical sheet is thus divided into five regions, which we denote successively by (1,1), (S_2), (1,2), (S_1), (2,2). The designations (1,1), (1,2) ... are chosen so that they are related to the nomenclature of the 1st and 2nd sheets in the previous paragraph, and indicate whether we have to calculate the considered region using the first or second sheet of the functions $u(\varphi')$ and $u(-\varphi')$, respectively; in (1,2), for example, the first number refers to $u(\varphi')$ and the second number refers to $u(-\varphi')$. Thus the approximation formulas that we have to apply in the different regions follow immediately. Namely, we have to use formula (10) on page 361 for $u(\varphi')$ in regions (1,2) and (1,2) and formula (9) for $u(\varphi')$ in region (2,2); while, on the other hand, we have to use formula (10) for $u(-\varphi')$ in region (1,1) and formula (9) for $u(-\varphi')$ in regions (1,2) and (2,2). If we proceed from the auxiliary function Z to the oscillatory component $Z = \mathfrak{Re}(e^{\frac{2\pi i t}{\tau}} Z)$, then we obtain for Z the following approximation formulas:

$$\text{in (2,2)}\quad \frac{1}{4\pi}\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left\{\frac{\pm 1}{\cos\frac{\varphi+\varphi'}{2}}-\frac{1}{\cos\frac{\varphi-\varphi'}{2}}\right\},$$

$$\text{in (1,2)}\quad \cos 2\pi\left(\frac{r}{\lambda}\cos(\varphi-\varphi')+\frac{t}{\tau}\right)+\frac{1}{4\pi}\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left\{\frac{\pm 1}{\cos\frac{\varphi+\varphi'}{2}}-\frac{1}{\cos\frac{\varphi-\varphi'}{2}}\right\},$$

*) Fig. 5 is based on the following proportions: $\lambda = 5 \cdot 10^{-4}$ mm (D-line), $\varepsilon = 3 \cdot 10^{-3}$. The boundary parabolas of S_1 and S_2 become so thin that they can only be indicated schematically. The distance of the vertex from the winding point is approximately $e = \frac{1}{300}$ mm. For the region S_2' (cf. page 372), there results $e' = 1$ mm.

$$\text{in (1,1)}\quad \cos 2\pi\left(\frac{r}{\lambda}\cos(\varphi-\varphi')+\frac{t}{\tau}\right)\mp\cos 2\pi\left(\frac{r}{\lambda}\cos(\varphi+\varphi')+\frac{t}{\tau}\right)$$
$$+\frac{1}{4\pi}\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\frac{\pi}{4}\right]\sqrt{\frac{\lambda}{r}}\left\{\frac{\pm 1}{\cos\frac{\varphi+\varphi'}{2}}-\frac{1}{\cos\frac{\varphi-\varphi'}{2}}\right\}.$$

Here the upper signs apply for case a) and the lower signs for case b).

Some interesting observations are connected with these formulas.

Let us consider first the *region* (2,2) (the region *of the geometric shadow*, as one usually says). The light motion here bears throughout the character that was described in the previous paragraph as the wave motion of the second sheet.

In the language of geometric optics*) one would interpret formula (2,2) in the following way: *rays propagate from the winding point* $r=0$ *in the direction of the radius vector throughout the entire interior of region* (2,2), *exactly as if the winding point were a luminous point.*

The oscillatory state along a single ray is given, as for a usual plane wave, by $\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)+\delta\right]$, where the term $\delta=\frac{\pi}{4}$ determines the phase of the light. The amplitude of the light decays with increasing r as $\sqrt{\frac{1}{r}}$, and thus in the same ratio as the light from a luminous point in a plane problem according to the interpretation of geometric optics. The amplitude is in general very small due to the factor $\sqrt{\lambda}$, and changes furthermore with the direction of the rays according to the law

$$\left\{\frac{\pm 1}{\cos\frac{\varphi+\varphi'}{2}}-\frac{1}{\cos\frac{\varphi-\varphi'}{2}}\right\}.$$

This is what geometrical optics would say, and our eye sees it in exactly the same way, as if our visual nerves were tuned precisely to the concepts of geometric optics. *Our eye actually attributes the rays to a light source that is supposed to be at the winding point;* that is, *the edge of the screen appears, as seen from the region of the geometric shadow, as a thin luminous line.* This is obviously an optical illusion. In truth there is no singularity at the winding

*) I apply these words in a somewhat extended sense to that entire way of thinking which decomposes the optical state into individual independently progressing rays.

point. The error occurs because the eye forms the analytic continuation of the approximation formula (2,2) across the transition parabola, which is not allowed. *The eye should not form the analytic continuation from the approximation formula, but rather from the exact formula.*

From the standpoint of geometric optics, the strange distribution of light in the region (2,2) is an exceptional state, whose appearance is called diffraction. We can, if we wish, accept this name and say: in the geometric shadow there is *diffracted* light.

Let us now pass into the *region* (1,2). Here our approximation formula separates into two terms. The first has exactly the form of a plane wave; it can also be written as

$$\cos 2\pi \left(\frac{x \cos \varphi' + y \sin \varphi'}{\lambda} + \frac{t}{\tau} \right).$$

This term represents that light which we wished to produce when we let so-called parallel light fall on the screen, but which we could not produce because of the presence of the screen. The second term has the same form as formula (2,2).

Geometric optics would speak in this region of two types of light, the incident and the diffracted light. It would decompose the first into rays that all run *parallel*, and the second into rays that *converge in the winding point. Our eye sees things in this same way.* If it accommodates itself at *infinity*, it recognizes only the light of the *first* term; if it accommodates itself at the *winding point*, it sees only the *second term*, and thinks that it sees the edge of the screen as luminous. Because in our theory the two terms separate from each other so smoothly, we can also assign them particular names and call the first *incident* and the second *diffracted* light.

In *region* (1,1), finally, our approximation formula separates into three different terms. The first two have the form of usual parallel light. The parallel light of the first term has the incident direction $\varphi = \varphi'$, and that of the second term the direction $\varphi = -\varphi'$. Geometric optics would designate the first term as *incident*, the second as *reflected*, and the third as *diffracted* light. *In the process of vision, these three terms separate from each other in the same way as in our formulas.*

Fundamentally, our conception is evidently that our function is something *uniform*, just as the physical state that it represents, which is determined by the boundary conditions of the problem. If we choose particular parts of it and label them with particular names, this is only for convenience and hardly serves for the comprehension of the whole.

The formulas of pages 368, 369 are in essence precisely those of Mr. Poincaré*). The remarkable circumstance is that for Mr. Poincaré the original problem is completely different from ours. Namely, he imagines that light is concentrated by a lens on the edge of the screen, instead of falling from a plane wave at infinity as we assume. Furthermore, Mr. Poincaré calculates from the beginning with approximate formulas, in that he neglects higher powers of λ. In this way, however, the criterion for the region of applicability of the approximate formulas is lost, and there arises the difficulty**) that *singularities* appear in the shadow boundaries of the first and second kind (because of the factors $\frac{1}{\cos\frac{\varphi-\varphi'}{2}}$ and $\frac{1}{\cos\frac{\varphi+\varphi'}{2}}$, respectively), which are physically impossible. *Our theory provides a complete resolution of this difficulty.* For we know that the validity of the Poincaré formulas extends only to the *parabolas*, but not to the *axes of the parabolas*.[48]

We turn in conclusion to the transition strips, and in particular to the region S_1. Here we must return to the exact formula (5) on page 359. In order to obtain a connection with Kirchhoff's *Optik*, we wish to introduce the functions

$$M(x) = \int_x^\infty \cos(x^2-\tau^2)\,d\tau = \mathfrak{Re}\left(e^{ix^2}\int_x^\infty e^{-i\tau^2}\,d\tau\right),$$

$$N(x) = \int_x^\infty \sin(x^2-\tau^2)\,d\tau = \mathfrak{Im}\left(e^{ix^2}\int_x^\infty e^{-i\tau^2}\,d\tau\right).$$

We can also write equation (5) on page 359 as

$$u = e^{-\frac{2\pi i r}{\lambda}+\frac{i\pi}{4}}\frac{1}{\sqrt{\pi}}\int_{-T}^{+\infty} e^{-i(\tau^2-T^2)}\,d\tau. \tag{4}$$

If we form in cases a) and b) the function Z from equation (3), and furthermore the oscillatory component $Z = \mathfrak{Re}(e^{\frac{2\pi i t}{\tau}}\mathsf{Z})$, then we obtain

$$\begin{aligned} Z &= \frac{1}{\sqrt{\pi}}\cos\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)-\frac{\pi}{4}\right]\Big(M(-T_1)\mp M(-T_2)\Big)+ \\ &\quad+\frac{1}{\sqrt{\pi}}\sin\left[2\pi\left(\frac{r}{\lambda}-\frac{t}{\tau}\right)-\frac{\pi}{4}\right]\Big(N(-T_1)\mp N(-T_2)\Big), \\ T_1 &= \sqrt{\frac{4\pi r}{\lambda}}\cos\frac{\varphi-\varphi'}{2},\quad T_2 = \sqrt{\frac{4\pi r}{\lambda}}\cos\frac{\varphi+\varphi'}{2}. \end{aligned}$$

*) Poincaré: l. c. p. 318.

**) l. c. pag. 314.

These equations give the exact expression for the oscillatory state *at an arbitrary time*, and indeed not only in the transition strip, but rather at all points of the Riemann surface. We compute from these the *mean value of the oscillatory energy over the duration of an oscillation*, the so-called "*intensity of light*"; that is,

$$J = \frac{1}{\tau}\int_0^\tau Z^2\, d\tau.$$

There results simply

(5) $\frac{1}{2\pi}(\mathfrak{M}^2 + \mathfrak{N}^2)$, $\mathfrak{M} = M(-T_1) \mp M(-T_2)$, $\mathfrak{N} = N(-T_1) \mp N(-T_2)$.

We can replace this expression by a simpler one in the largest part of the transition strip S_1 without affecting the prescribed precision. We first separate from S_1 that part which also belongs to S_2. In the remaining part $(S_1 - S_2)$, we can replace $M(-T_2)$ and $N(-T_2)$ by approximate formulas. Namely, if we expand the integral appearing in (4) according to equation (6) on page 359, then we may terminate the expansion at the first term and obtain

$$M(-T_2) = 0, \quad N(-T_2) = \frac{1}{2\sqrt{\pi}}\frac{1}{|T_2|}.$$

Furthermore, we delimit outside the parabola which bounds the region S_2, and which corresponds to the precision ε, a second parabola, which corresponds to the precision ε^2 and which bounds the region S_2'. Its dimensions are determined according to equation (8) on page 360 by the quantity $e' = \frac{\lambda}{|16\pi\varepsilon^2|}$. The inequality $\frac{1}{4T^2} < \varepsilon^2$ or $\frac{1}{2|T|} < \varepsilon$, analogous to (7) on page 360, holds at all points that lie outside this parabola. If we also separate from S_1 this region S_2', then $N(-T_2)$ becomes appreciably zero for our purpose in the remaining part $S_1 - S_2'$. Thus, the expression for the intensity reduces to

(6) $$2\pi J = M^2(-T_1) + N^2(-T_1),$$

which correctly represents the true value in the region $S_1 - S_2'$ up to an allowed error.

We compare this with the expression

(7) $$CJ' = M^2(u) + N^2(u)$$

given by Kirchhoff*).[49] Here C denotes a constant that remains undetermined, which is taken into the unit of measure of light intensity and which

*) Kirchhoff: Math. Optik, 7^te Vorlesung §3.

we can ignore. The quantity u is defined by equation (11a) of Kirchhoff. If, in order to arrive at the conditions and notation of our problem, we set in Kirchhoff's formulas

$$z_1 = \infty, \quad x_0 = r\cos(\varphi - \varphi' + \pi), \quad z_1 = r\sin(\varphi - \varphi' + \pi),$$

then there follows

$$u^2 = \frac{r\pi}{\lambda}\frac{\sin^2(\varphi - \varphi' + \pi)}{\cos(\varphi - \varphi' + \pi)},$$

while for us

$$T_1^2 = \frac{4r\pi}{\lambda}\cos^2\frac{\varphi - \varphi'}{2}.$$

These two quantities become identical for the boundary of the geometric shadow of the first kind. In fact, if we expand them in increasing powers of the quantity $\vartheta = \varphi - \varphi' + \pi$, which one calls the *diffraction angle*, then we obtain the series

$$u^2 = \frac{r\pi}{\lambda}\left(\vartheta^2 + \frac{\vartheta^4}{6} + \cdots\right),$$

$$T_1^2 = \frac{r\pi}{\lambda}\left(\vartheta^2 - \frac{\vartheta^4}{12} + \cdots\right),$$

which coincide exclusively up to terms of the 4^{th} order. We conclude from this that we can delimit in the vicinity of the shadow boundary a nonzero finite region in which the quantities T_1 and u differ by less than an arbitrary finite quantity. Since expressions (6) and (7) are continuous functions of T_1 and u, respectively, we may further say that we can delimit a nonzero finite region in the neighborhood of the shadow boundary in which the difference between the Kirchhoff formula and our formula lies beneath the allowed error bound. Finally, since our formula (6) differs by less than ε from the exact formula at a sufficient distance from the winding point (that is, at such points that lie outside S_2'), we can go further: *the Kirchhoff formula gives the true value of the intensity with sufficient precision at all those points that 1) lie sufficiently close to the shadow boundary and 2) are sufficiently far from the winding point.*

We thus come to the remarkable result that we can confirm the conclusions of the older diffraction theory to a certain extent, while we must declare the method by which they were derived as completely inadmissible. The conditions that were just given for the validity of the Kirchhoff formula also exist in the older diffraction theory, but they could not, in that theory, be made precise numerically as here. In fact,

the older diffraction theory only makes the claim that it can represent the diffraction phenomena correctly in the immediate vicinity of the shadow boundary and at a very large distance from the edge of the screen.

The region of validity of the Kirchhoff formulas is accordingly only very small; outside of this region they become noticeably false. In this case, rather, the *Poincaré formulas* become correct. In that our theory lets the one and the other appear as appropriate approximations of the exact formulas, it forms the bridge between those two theories, and assigns to each its limited region of validity.

We can now dispose of the question concerning the *experimental verification* of our theory with a word. On one hand, the Kirchhoff formulas have been verified repeatedly for observations under a small angle of diffraction; on the other hand, Mr. Poincaré compares his formulas with the observations of Gouy under large diffraction angles and finds in essence a confirmation of them. *To the same extent, that is, for small as well as for large diffraction angles, our theory is therefore confirmed by experience.*

Particularly appropriate for a comparison with our theory are the observations of Mr. Maey,[50] which are distinguished by the simplicity of the experimental arrangement and the precision of the quantitative measurements*). This as well as further physical consequences I think to address in a different place. Furthermore, I hope soon to be able to report solutions of the differential equation $\Delta u + k^2 u = 0$ that are single-valued on a Riemann surface with two or infinitely many branch points in a finite region.[51] These would then allow for an exact treatment of diffraction phenomena that are caused by a slit or a grating.

Göttingen, summer 1895.

*) Cf. Wiedemann's Annalen 49, 1893.

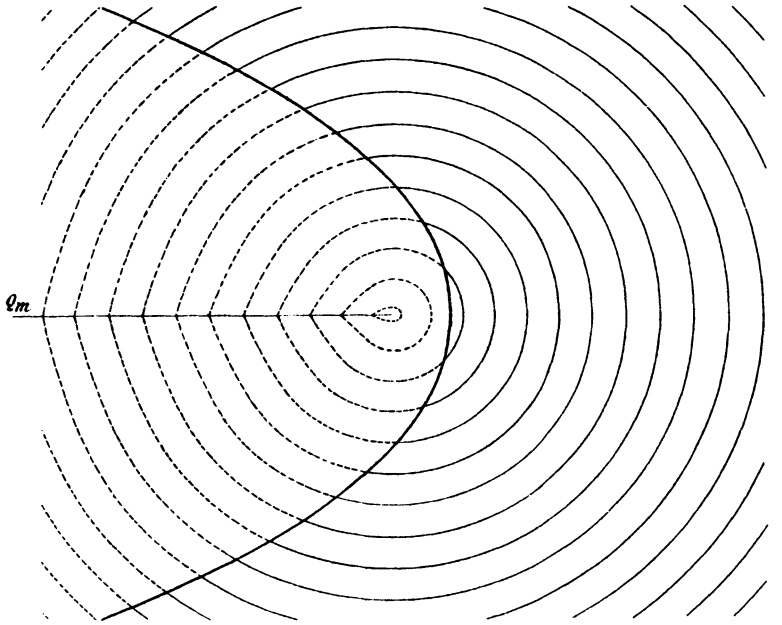


Fig. 1

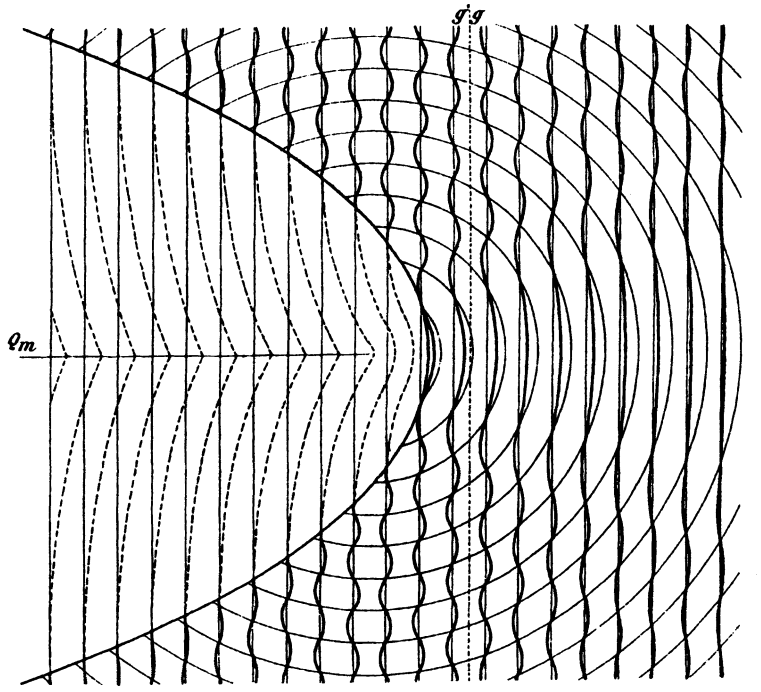


Fig. 2.

Sommerfeld (seated second from the left) in Göttingen, 1895.

Translators' Notes

1. (page 319) The conceptual formation of a Riemann double-space is illustrated in Fig. 6. Two identical realizations (*Exemplare*) of three-dimensional space are cut and opened along the surface S, so that each realization of S has an upper and a lower surface. To form the Riemann double-space, the two spaces are joined by identifying the upper surface S_1^+ of the first realization with the lower surface S_2^- of the second realization and identifying the lower surface S_1^- of the first realization with the upper surface S_2^+ of the second realization.

2. (page 319) The geometric configuration described by Sommerfeld is shown in Fig. 7. The two realizations of space are cut, opened, and joined along S as in Fig. 6. The use of auxiliary sources in unphysical regions of space for the solution of boundary value problems is now called the method of images.

In the present paper, Sommerfeld never addresses the three-dimensional problem that he describes here. In the subsequent paper "Über verzweigte Potentiale im Raum" (*Proceedings of the London Mathematical Society*, Vol. 28, 1897, pp. 395–429), Sommerfeld derives the three-dimensional electrostatic field due to a unit point charge at P when the screen S is a half plane with its edge parallel to the z-axis and when the screen S is an infinite strip bounded by two lines parallel to the z-axis. See also a correction to this paper by Sommerfeld in the *Proceedings of the London Mathematical Society*, Vol. 30, 1899, pp. 161–163.

Some exact solutions of Sommerfeld's three-dimensional diffraction problem are given by H. S. Carslaw ("Some Multiform Solutions of the Partial Differential Equations of Physical Mathematics and their Applications, " *Proceedings of the London Mathematical Society*, Vol. 30, 1899, pp. 121–161 and "Oblique Incidence of a train of Plane Waves on a Semi-Infinite Plane," *Proceedings of the Edinburgh Mathematical Society*, Vol. 19, 1901, pp. 71–77).

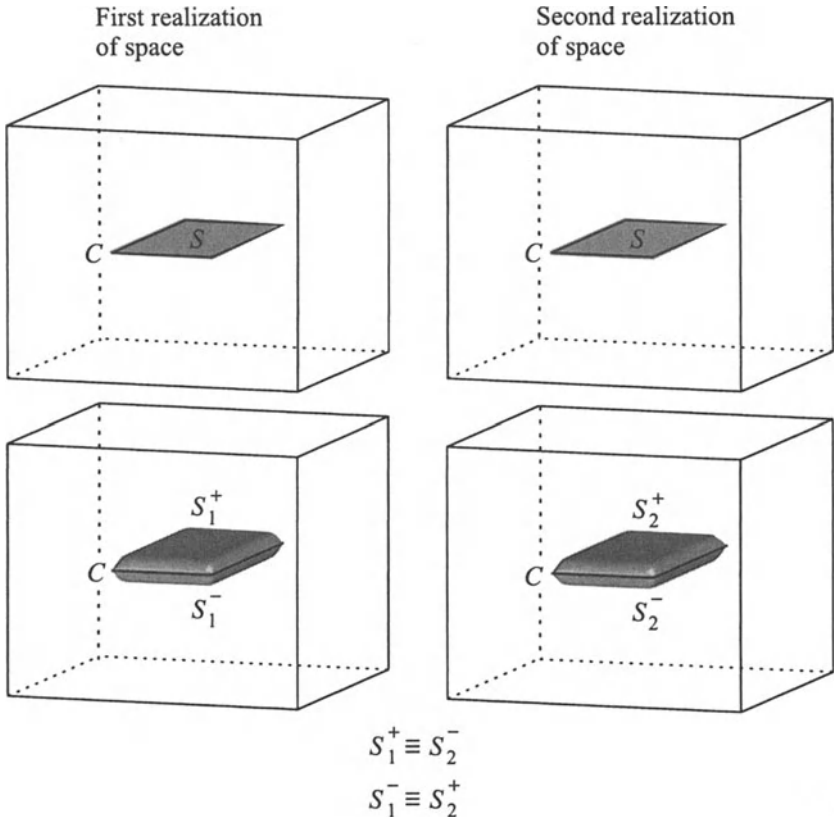


FIG. 6. Conceptual formation of a Riemann double-space.

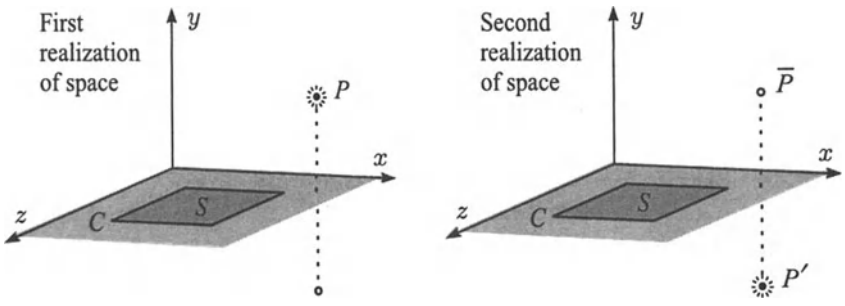


FIG. 7. Geometric configuration for Sommerfeld's three-dimensional diffraction problem.

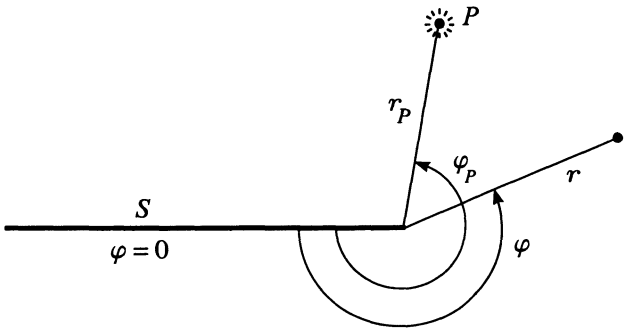


FIG. 8. Electrostatic point charge P in the presence of a perfectly conducting screen S.

Carslaw assumes that the screen S in Fig. 7 is the half plane $y = 0$, $x > 0$. He derives the fields due to both a point source P and an incident plane wave whose direction of propagation may be oriented arbitrarily in space.

3. (page 320) To illustrate the solution of the two-dimensional Laplace equation $\Delta u = 0$ in a context that is closely related to Sommerfeld's diffraction problem, we consider the two-dimensional electrostatic field of a unit point charge P in the presence of a perfectly conducting screen S. The charge has coordinates (r_P, φ_P) with respect to the (r, φ) polar coordinate system defined in Fig. 8. The screen S occupies the line $\varphi = 0$. If we introduce the complex potential $v = \phi + i\psi$, where ϕ is the real electrostatic potential and $\psi = \text{constant}$ is a line of electrostatic force, then v satisfies the Laplace equation $\Delta v = 0$ with the conditions $\phi = 0$ for $\varphi = 0$ and $\phi = \infty$ as $\log \frac{1}{R}$ at P, where R denotes the distance from P.

To determine the function v, we first consider the complex potential $u(P)$ of a unit point charge P on the two-sheeted Riemann surface shown in Fig. 9(a). The Riemann surface is formed by cutting and joining two planes, and is parameterized by polar coordinates (r, φ), with $0 \leq \varphi < 2\pi$ on the first sheet and $2\pi \leq \varphi < 4\pi$ on the second sheet. The coordinates of the charge P on the Riemann surface are (r_P, φ_P). The function $u(P)$ again satisfies the Laplace equation

$$\Delta u(P) = 0, \tag{1}$$

now defined on the two-sheeted Riemann surface, with only the condition

$$u(P) = \infty \text{ as } \log \frac{1}{R} \text{ at } P.$$

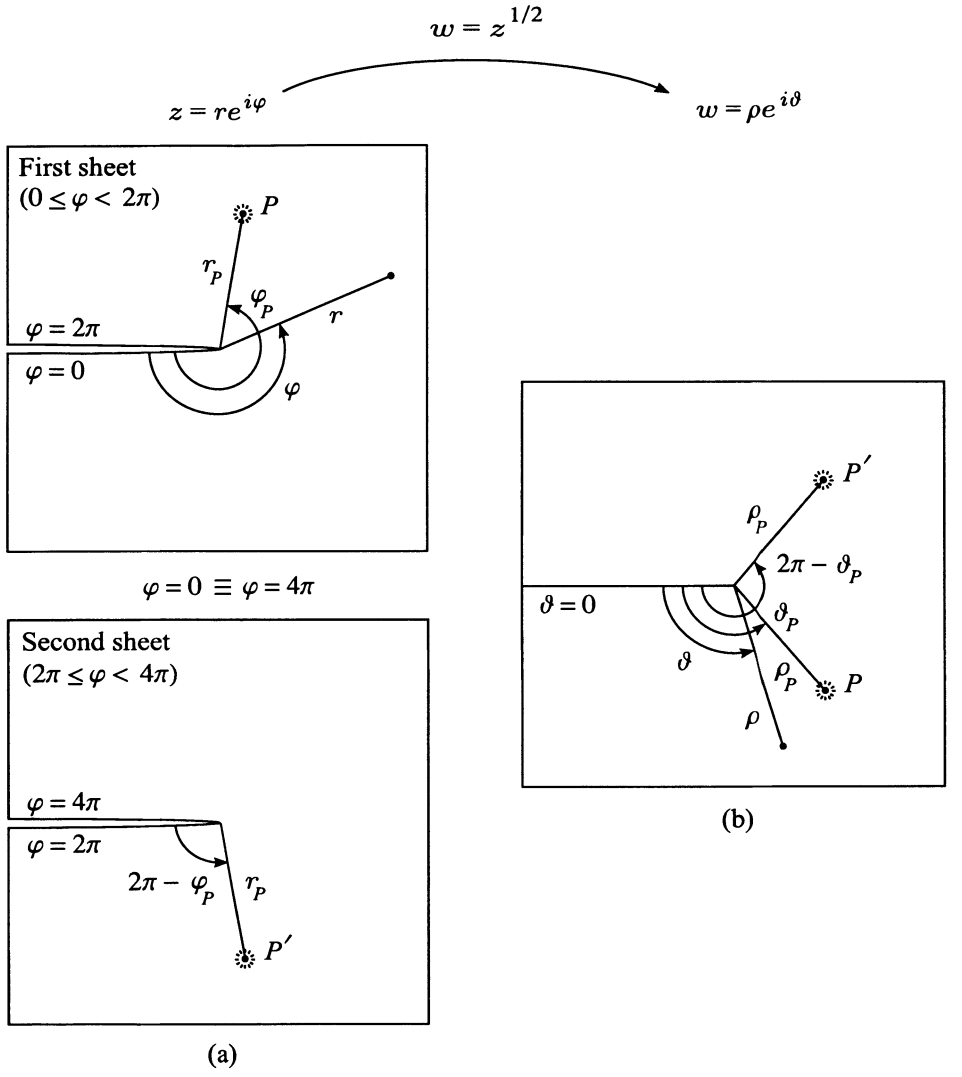


FIG. 9. Two-sheeted Riemann surface and smooth plane used in the solution of the electrostatic boundary value problem.

The two-sheeted Riemann surface of Fig. 9(a) can be placed into a one-to-one correspondence with the smooth plane in Fig. 9(b) by a conformal map. If the points of the Riemann surface are represented by the complex numbers $z = re^{i\varphi}$ and the points of the smooth plane are represented by the complex numbers $w = \rho e^{i\vartheta}$, then the conformal map that relates the Riemann surface and the plane is

$$w = z^{1/2}, \quad \text{or} \quad \rho = \sqrt{r} \quad \text{and} \quad \vartheta = \varphi/2. \tag{2}$$

The Laplace equation (1) is invariant under the conformal map, and the function $u(P)$ on the ordinary plane is

$$u(P) = \frac{1}{2\pi} \log(w - w_P) = \frac{1}{2\pi} \log(\rho e^{i\vartheta} - \rho_P e^{i\vartheta_P}).$$

Using the conformal map relations (2), the function $u(P)$ on the two-sheeted Riemann surface is

$$u(P) = \frac{1}{2\pi} \log(\sqrt{r}\, e^{i\varphi/2} - \sqrt{r_P}\, e^{i\varphi_P/2}).$$

To obtain the complex potential v for the point charge in Fig. 8, we now form the difference

$$v = u(P) - u(P'),$$

where P' is the image point with coordinates $(r_P, 4\pi - \varphi_P)$ shown in Fig. 9. Thus,

$$v = \frac{1}{2\pi} \log(\sqrt{r}\, e^{i\varphi/2} - \sqrt{r_P}\, e^{i\varphi_P/2}) - \frac{1}{2\pi} \log(\sqrt{r}\, e^{i\varphi/2} - \sqrt{r_P}\, e^{i(4\pi-\varphi_P)/2}).$$

Because of the symmetric arrangement of P and P' in Fig. 9(a), the real part of v vanishes on the lines $\varphi = 0$ and $\varphi = 2\pi$ (but not on the lines $\varphi = \pi$ and $\varphi = 3\pi$). The real part of the function $v(r, \varphi)$ in the domain $0 \leq \varphi \leq 2\pi$ is thus the electrostatic potential for the original physical problem. Some equipotentials (lines of constant $\phi = \mathrm{Re}\,(v)$) and lines of electrostatic force (lines of constant $\psi = \mathrm{Im}\,(v)$) are shown in Fig. 10 for the case $\varphi_P = 5\pi/4$.

The function $u(P)$ on the two-sheeted Riemann surface is interesting in itself. In Fig. 11, we show equipotentials and lines of force for $u(P)$ when the charge P is on the line $\varphi_P = 5\pi/4$. Fig. 11(a) shows equipotentials and lines of force on the first sheet ($0 \leq \varphi < 2\pi$) of the Riemann surface, and Fig. 11(b) shows equipotentials and lines of force on the second sheet ($2\pi \leq \varphi < 4\pi$). Figs. 12(a) and 12(b) show the analogous results for $\varphi_P = \pi$ and the same value of r_P as in Fig. 11. An equipotential on the first sheet in Fig. 11(a) that

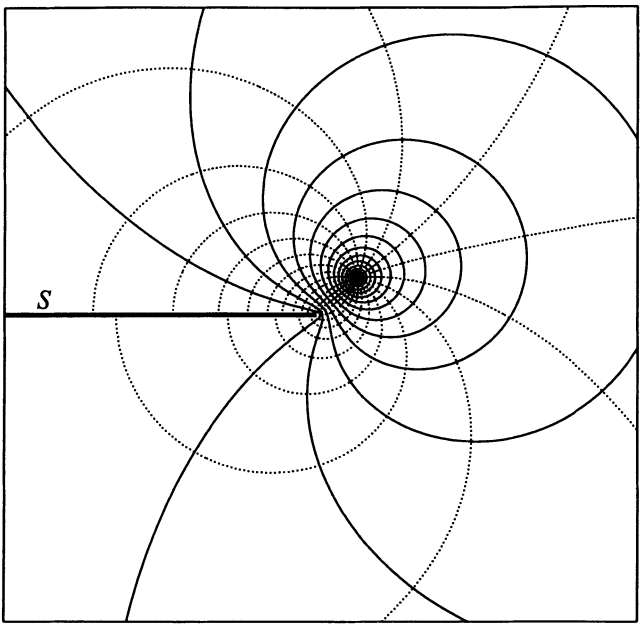


FIG. 10. Equipotentials (—) and lines of electrostatic force (···) for a point charge in the presence of a perfectly conducting screen S.

crosses the line $\varphi = 2\pi$ passes onto the second sheet in Fig. 11(b), curves around the origin, and rejoins itself on the first sheet at $\varphi = 0$. If such an equipotential is projected onto a plane, we obtain the closed curve shown in Fig. 13(a). The equipotential does not intersect itself on the two-sheeted Riemann surface, but loops once around the winding point (*Windungspunkt*) at the origin. Fig. 13(b) shows the corresponding projection for an equipotential of Fig. 12. Since the real part of $u(P)$ depends on the angles φ and φ_P only through the difference $\varphi - \varphi_P$, the equipotentials in Figs. 11 and 12 are related by a simple rotation.

The method used to solve this electrostatic problem consists of (i) reformulating the problem on a Riemann surface; (ii) deriving the function $u(P)$ for the potential of a point charge at position P on the Riemann surface; (iii) forming the difference $v = u(P) - u(P')$, where P' is the image point of P on the Riemann surface with respect to the screen S. Sommerfeld uses exactly these steps in the solution of his diffraction problem. Sommerfeld cannot, however, use a simple conformal map to derive the appropriate function $u(P)$ for the diffraction problem, but must invent a completely new procedure.

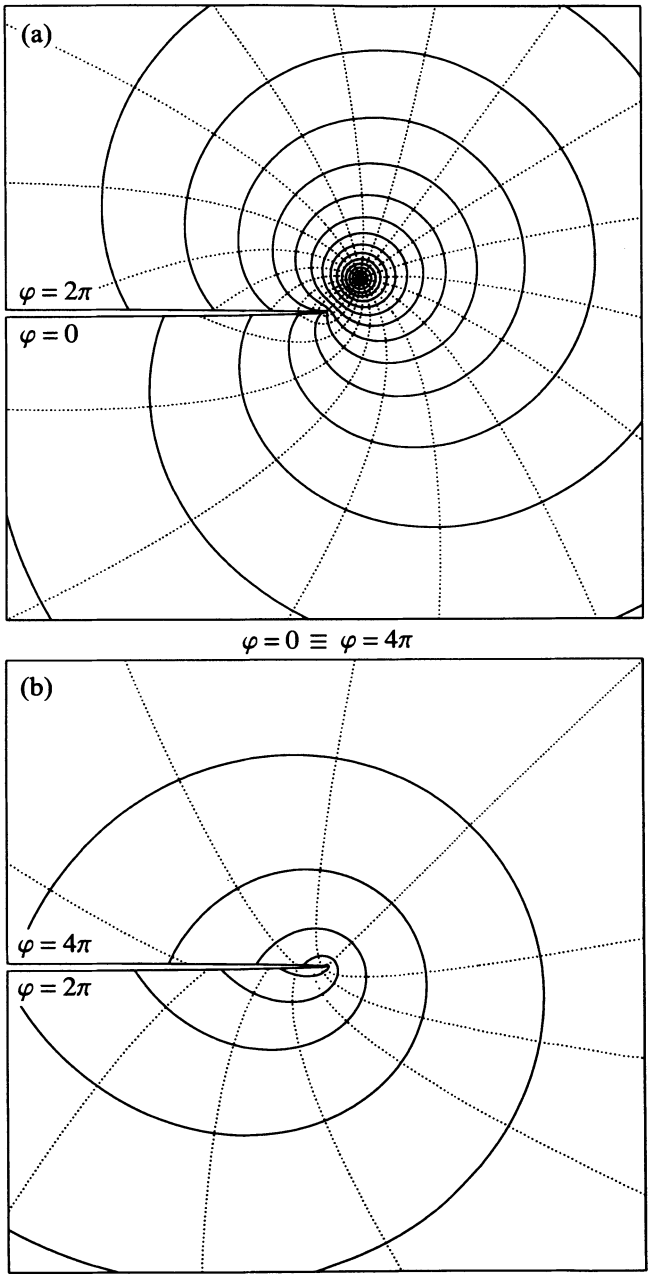


FIG. 11. Equipotentials (—) and lines of electrostatic force force (···) for the function $u(P)$ on the Riemann surface for $\varphi_P = 5\pi/4$.
(a): first sheet; (b): second sheet.

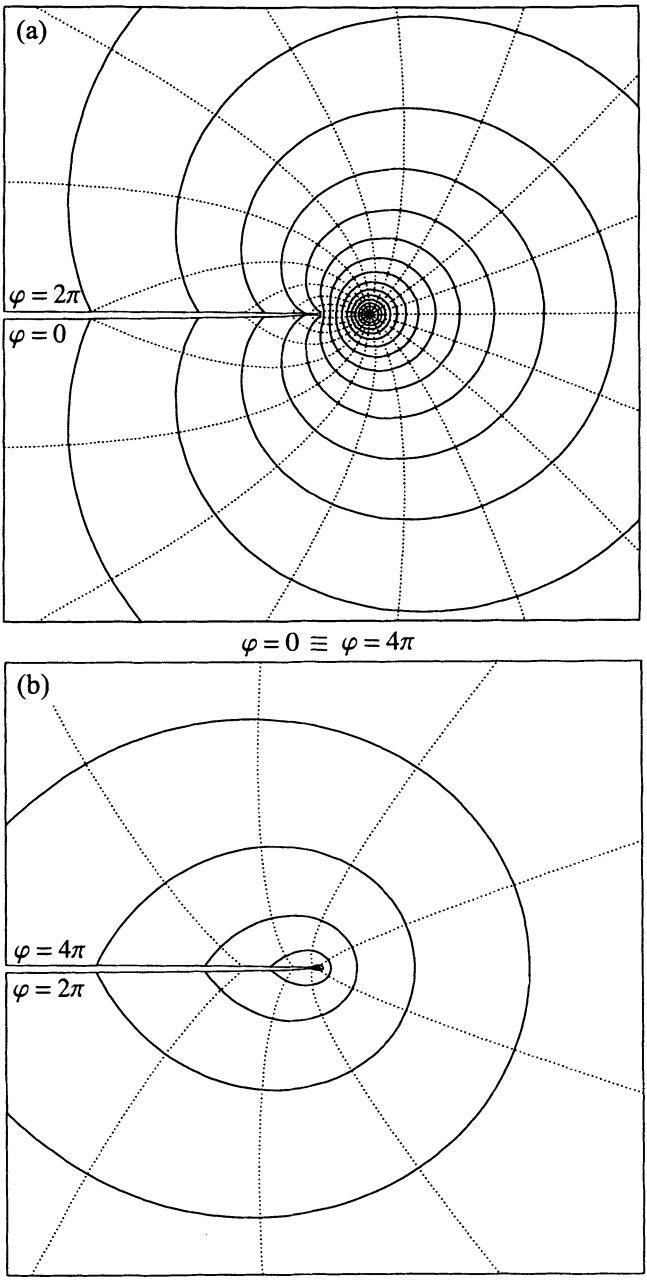


FIG. 12. Equipotentials (—) and lines of electrostatic force (···) for the function $u(P)$ on the Riemann surface for $\varphi_P = \pi$.
(a): first sheet; (b): second sheet.

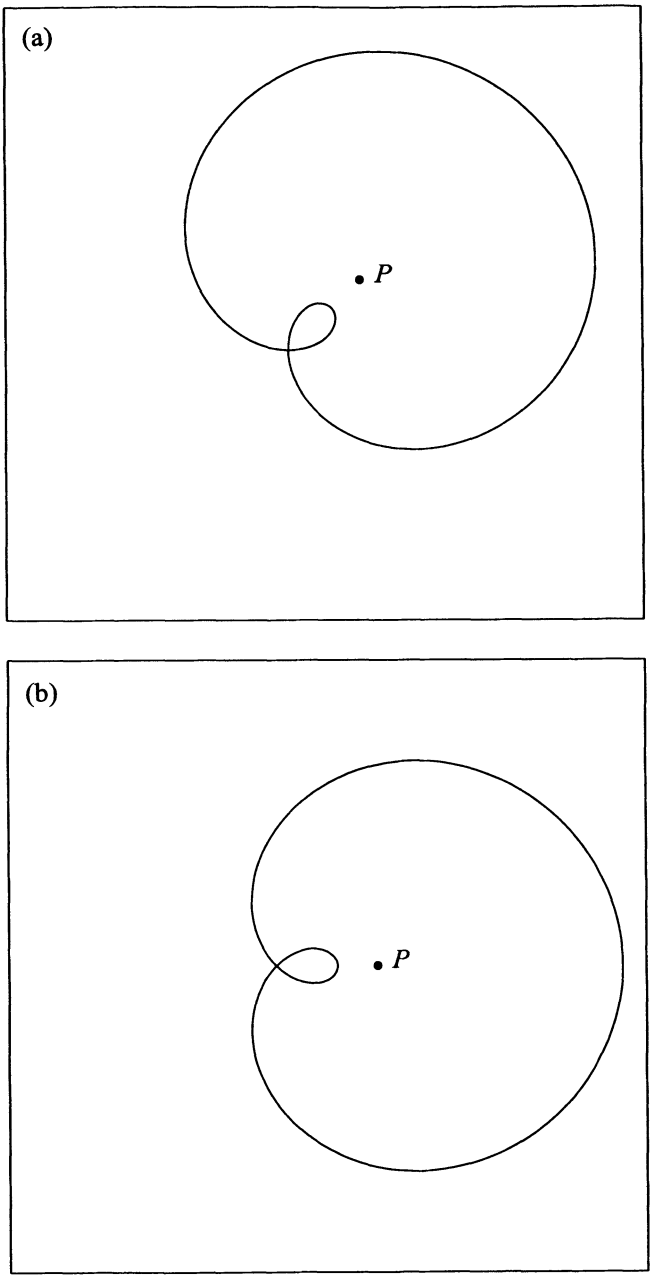


FIG. 13. Plane projections of equipotentials of the function $u(P)$.
(a): $\varphi_P = 5\pi/4$; (b): $\varphi_P = \pi$.

4. (page 327) In current notation, Sommerfeld's function $U_\nu(x)$ is

$$U_\nu(x) = \frac{\pi}{2i} H_\nu^{(1)}(x),$$

where $H_\nu^{(1)}$ is the Hankel function of the first kind of order ν. The function $K_n(x)$ that Sommerfeld calls Heine's Bessel function of the second kind is

$$K_n(x) = \frac{\pi}{2} Y_n(x),$$

where $Y_n(x)$ is the Bessel function of the second kind that is used in the modern English literature. A summary of the various forms that have been used for the Bessel function of the second kind is given by G. N. Watson in *A Treatise on the Theory of Bessel Functions* (Cambridge University Press, 2nd ed., 1995, §3.5–3.582, pp. 57–73).

5. (page 328) Fig. 14 shows the integration path in the α plane that corresponds to the z plane path in Sommerfeld's Fig. 1. The parameter ρ is the radius of the semicircle in the z plane. The integration path in Fig. 14 is a special case of Sommerfeld's path (B) in §4.

6. (page 330) The stereographic projection from the South pole of the unit sphere to the complex Z plane is illustrated in Fig. 15.

7. (page 331) According to E. T. Whittaker and G. N. Watson in *A Course of Modern Analysis* (Cambridge University Press, 1945, p. 403), William Fishburn Donkin (1814–1869) proved in 1857 that the general spherical harmonic V of order zero is

$$V = f\left(\frac{\xi + i\eta}{\zeta + \rho}\right) + F\left(\frac{\xi - i\eta}{\zeta + \rho}\right),$$

where f and F are arbitrary functions. Donkin was the Savilian Professor of Astronomy at Oxford University, and the subject of a photograph by the Oxford mathematician Charles Lutwidge Dodgson (Lewis Carroll).

8. (page 331) Maxwell's derivation of the spherical harmonics is discussed by R. Courant and D. Hilbert in *Methods of Mathematical Physics* (Interscience Publishers, 1965, Vol. I, pp. 514–521).

9. (page 333) The double loop (*Doppelschlinge*) is a contour C for which the bracketed terms in the footnote on page 332 resume their original values when C is described.

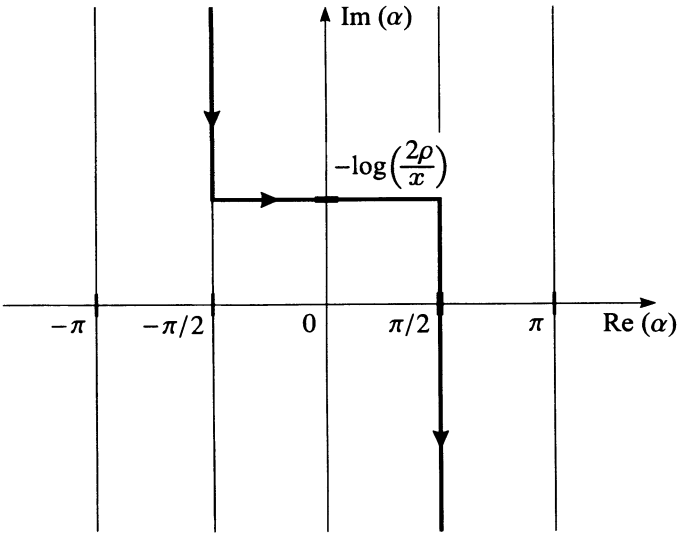


FIG. 14. Integration path in the α plane for the function $U_\nu(x)$.

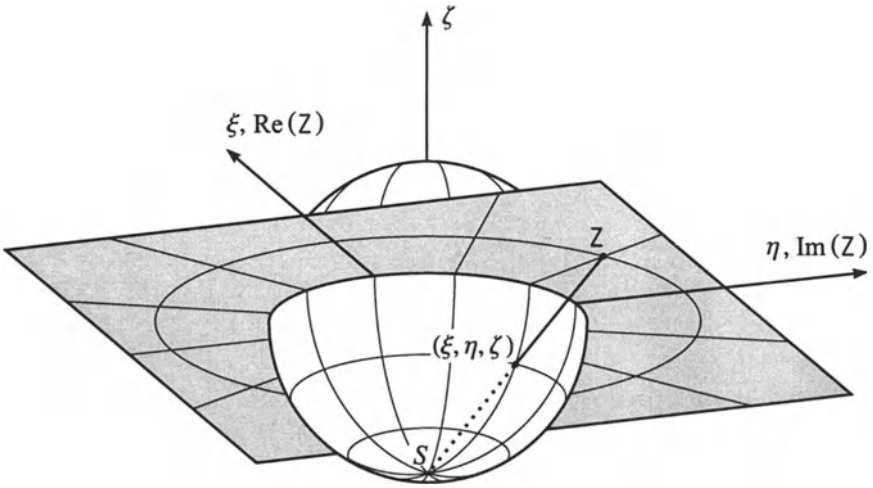


FIG. 15. Stereographic projection from the South pole of the unit sphere to the complex Z plane.

10. (page 333) The differential equation for the spherical surface function $u_m(\theta, \varphi)$ of order m is

$$\frac{1}{\sin^2\theta}\frac{\partial^2 u_m}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u_m}{\partial\theta}\right) + m(m+1)u_m = 0,$$

or

$$\frac{1}{\sin^2\theta}\frac{\partial^2 u_m}{\partial \varphi^2} + \frac{1}{\sin\theta}\frac{\partial}{\partial(\sin\theta)}\left(\sin\theta\frac{\partial u_m}{\partial(\sin\theta)}\cos\theta\right)\cos\theta + m(m+1)u_m = 0, \tag{3}$$

where the spherical polar angles θ and φ are related to Sommerfeld's rectangular coordinates ξ, η, ζ on the unit sphere by

$$\begin{aligned} \xi &= \sin\theta\cos\varphi \\ \eta &= \sin\theta\sin\varphi \\ \zeta &= \cos\theta. \end{aligned}$$

If we change independent variables from (θ, φ) to (r, φ), where

$$\sin\theta = \frac{kr}{m}, \quad \cos\theta = \sqrt{1 - \frac{k^2r^2}{m^2}},$$

then equation (3) becomes

$$\frac{1}{r^2}\frac{\partial^2 u_m}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_m}{\partial r}\sqrt{1-\frac{k^2r^2}{m^2}}\right)\sqrt{1-\frac{k^2r^2}{m^2}} + \frac{m(m+1)k^2}{m^2}u_m = 0.$$

If we now let $m \to \infty$, we obtain

$$\frac{1}{r^2}\frac{\partial^2 u_\infty}{\partial\varphi^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_\infty}{\partial r}\right) + k^2 u_\infty = 0,$$

which is the two-dimensional Helmholtz equation $\Delta u_\infty + k^2 u_\infty = 0$ in polar coordinates (r, φ).

11. (page 334) The last equation in Sommerfeld's (11) on page 334 is based on the identity

$$\lim_{m\to\infty}\frac{1}{(1-t/m)^{m+1}} = e^t.$$

This identity may be derived from the binomial theorem and an interchange of limits, as follows:

$$\begin{aligned}
&\lim_{m\to\infty} \frac{1}{(1-t/m)^{m+1}} = \\
&\quad = \lim_{m\to\infty} \left\{ \lim_{N\to\infty} \left[1 + \sum_{n=1}^{N} \frac{(-m-1)(-m-2)\cdots(-m-n)}{n!} \frac{(-t)^n}{m^n} \right] \right\} \\
&\quad = \lim_{N\to\infty} \left\{ \lim_{m\to\infty} \left[1 + \sum_{n=1}^{N} \frac{(-m-1)(-m-2)\cdots(-m-n)}{n!} \frac{(-t)^n}{m^n} \right] \right\} \\
&\quad = \lim_{N\to\infty} \left\{ 1 + \sum_{n=1}^{N} \frac{(-m)^n}{n!} \frac{(-t)^n}{m^n} \right\} \\
&\quad = \lim_{N\to\infty} \left\{ 1 + \sum_{n=1}^{N} \frac{t^n}{n!} \right\} \\
&\quad = e^t.
\end{aligned}$$

12. (page 334) To show that the function

$$u(r,\varphi) = \int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, e^{ikr\cos\alpha}\, d\alpha$$

is a solution of the differential equation

$$\Delta u + k^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0,$$

we note that

$$\begin{aligned}
\frac{\partial u}{\partial \varphi} &= \int \frac{\partial}{\partial \varphi} \left[f(e^{i(\varphi+\alpha-\frac{\pi}{2})}) \right] e^{ikr\cos\alpha}\, d\alpha \\
&= \int \frac{\partial}{\partial \alpha} \left[f(e^{i(\varphi+\alpha-\frac{\pi}{2})}) \right] e^{ikr\cos\alpha}\, d\alpha.
\end{aligned}$$

Integrating by parts with the assumption that the boundary terms vanish,

$$\frac{\partial u}{\partial \varphi} = -\int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, (-ikr\sin\alpha) e^{ikr\cos\alpha}\, d\alpha.$$

Proceeding in the same way for the evaluation of the second derivative of u with respect to φ, we obtain

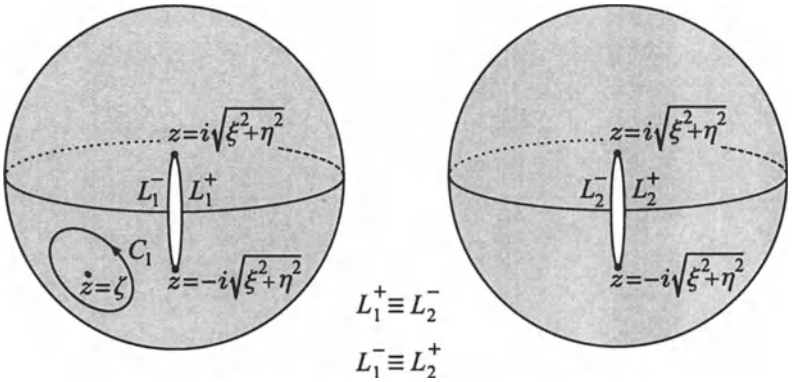
$$\begin{aligned}\Delta u + k^2 u &= \int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, e^{ikr\cos\alpha}(-k^2\cos^2\alpha)\, d\alpha + \\ &+ \frac{1}{r}\int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, e^{ikr\cos\alpha}(ik\cos\alpha)\, d\alpha + \\ &+ \frac{1}{r^2}\int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, e^{ikr\cos\alpha}(-ikr\cos\alpha - k^2r^2\sin^2\alpha)\, d\alpha + \\ &+ k^2\int f(e^{i(\varphi+\alpha-\frac{\pi}{2})})\, e^{ikr\cos\alpha}\, d\alpha \\ &= 0.\end{aligned}$$

13. (page 336) The integrand

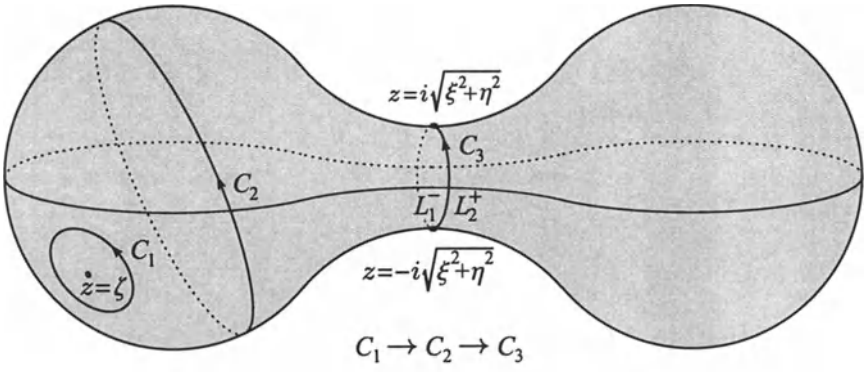
$$\frac{1}{(\zeta - z)^{m+1}\rho}$$

is single-valued on a Riemann surface of two sheets with branch points at $z = \pm i\sqrt{\xi^2+\eta^2}$. This surface may be constructed by first considering two complex z planes, each of which is cut from $z = -i\sqrt{\xi^2+\eta^2}$ to $z = i\sqrt{\xi^2+\eta^2}$. If each of these cut planes is mapped onto a sphere by stereographic projection, we obtain the two cut spheres shown in Fig. 16(a). The Riemann surface is formed by identifying the line L_1^- with L_2^+ and identifying the line L_1^+ with L_2^-. The resulting surface, which is topologically equivalent to a sphere, is illustrated in Fig. 16(b). A good introductory discussion of the topological properties of Riemann surfaces is given by G. Springer in *Introduction to Riemann Surfaces* (Addison-Wesley, 1957, Chapter 1).

The original closed integration path around $z = \zeta$ is denoted in Fig. 16(b) by C_1. Since there are no singularities between the pole at $z = \zeta$ and the branch points at $z = \pm i\sqrt{\xi^2+\eta^2}$, the integration path C_1 may be deformed across the point $z = \infty$ from C_1 to C_2 to C_3. The path C_3 surrounds the two sides of the original cut from $z = -i\sqrt{\xi^2+\eta^2}$ to $z = i\sqrt{\xi^2+\eta^2}$.



(a)



(b)

FIG. 16. Integration paths on a Riemann surface with branch points at $z = \pm i\sqrt{\xi^2 + \eta^2}$.

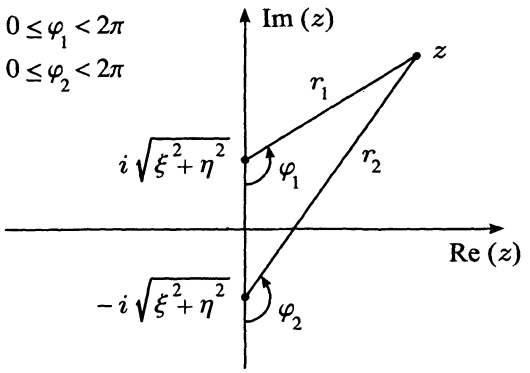


FIG. 17. Bipolar coordinates $(r_1, \varphi_1, r_2, \varphi_2)$ for the computation of $\rho = (\xi^2 + \eta^2 + z^2)^{1/2}$.

14. (page 336) The change of variables $z = i\sqrt{\xi^2 + \eta^2}\cos\alpha$ is equivalent to

$$\alpha = \cos^{-1}\frac{z}{i\sqrt{\xi^2+\eta^2}} = i\log\left[\frac{z+\rho}{i\sqrt{\xi^2+\eta^2}}\right], \quad \rho = (\xi^2+\eta^2+z^2)^{1/2}. \quad (4)$$

It is natural to consider equation (4) as a mapping between the complex α plane and the two-sheeted Riemann surface of Fig. 16. In order to compute ρ in a manner that is consistent with the cuts in Fig. 16(a), we introduce the bipolar coordinates $(r_1, \varphi_1, r_2, \varphi_2)$ in Fig. 17 and write

$$\rho = \pm\sqrt{r_1 r_2}\, e^{i(\varphi_1+\varphi_2-\pi)/2}.$$

The plus sign is used for points on the left half of the Riemann surface in Fig. 16(b), and the minus sign is used for points on the right half. The two cut z planes that are used in the construction of the Riemann surface are thus designated as the ρ^+ and ρ^- sheets of the Riemann surface.

The mapping (4) between the complex α plane and the two-sheeted Riemann surface of Fig. 16 is illustrated in Figs. 18 and 19. Points on the ρ^+ sheet in Figs. 19 and 20 are mapped into the upper half of the α plane, while points on the ρ^- sheet are mapped into the lower half of the α plane. The logarithm function in equation (4) produces a periodic repetition with period 2π along the direction of the real axis in the α plane. The contour C_3 in Fig. 16(b) is mapped into a segment of length 2π along the real axis of the α plane.

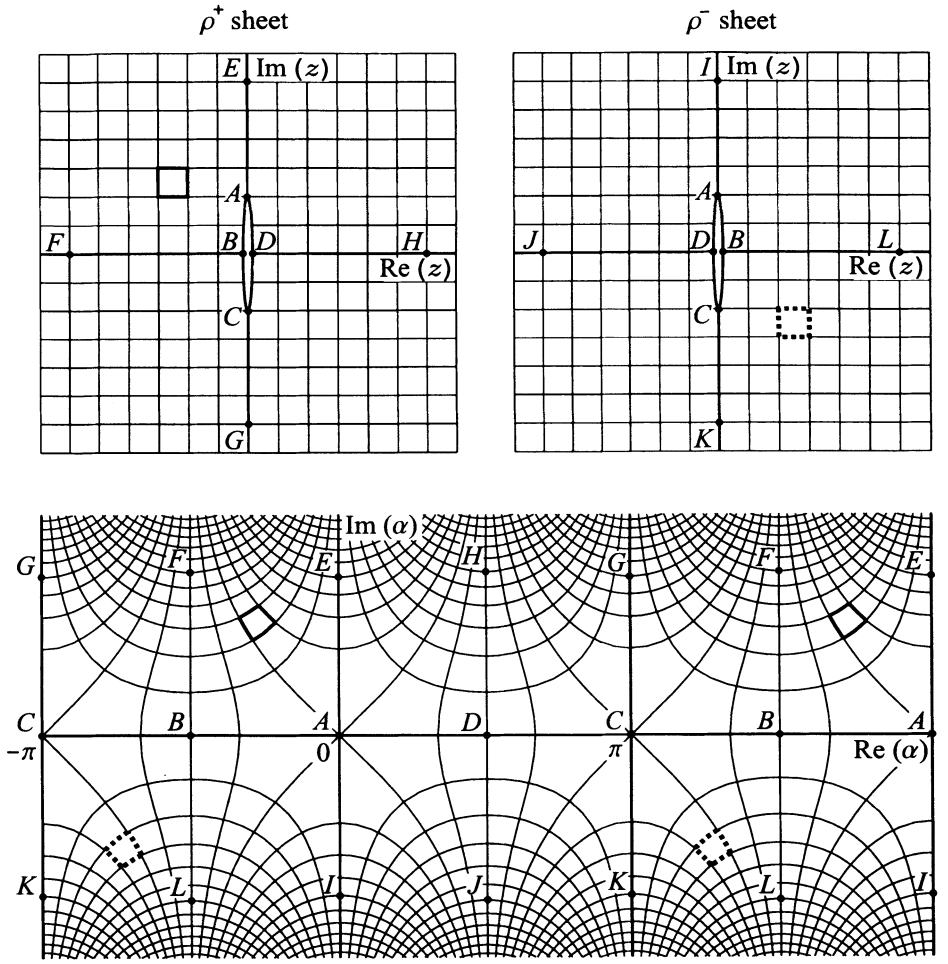


FIG. 18. Mapping $z = i\sqrt{\xi^2 + \eta^2}\cos\alpha$ between a two-sheeted Riemann surface and the complex α plane.

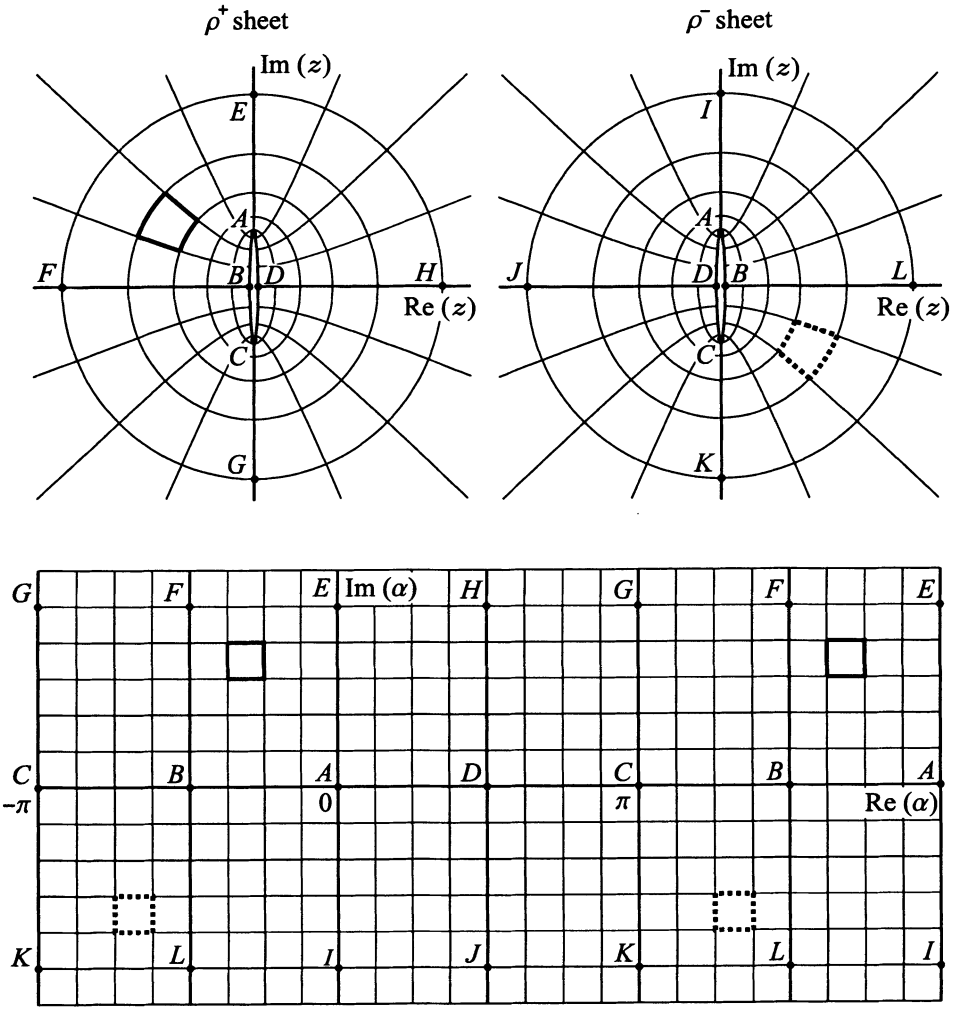


FIG. 19. Mapping $z = i\sqrt{\xi^2 + \eta^2}\cos\alpha$ between a two-sheeted Riemann surface and the complex α plane.

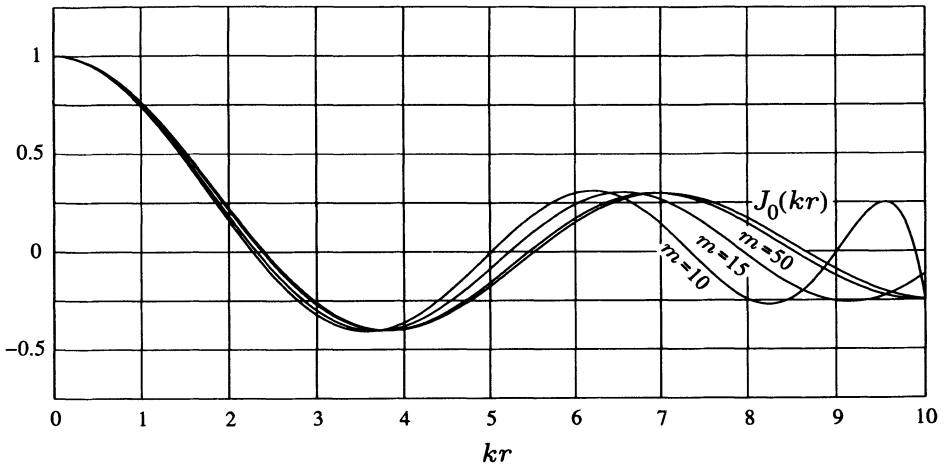


FIG. 20. Comparison of $J_0(kr)$ and $P_m\left(\sqrt{1-k^2r^2/m^2}\right)$ for increasing values of m.

15. (page 336) The Laplace integral representation of the zonal spherical functions is discussed by R. Courant and D. Hilbert, *loc. cit.*, pp. 501–504.

16. (page 336) The zonal spherical functions are $P_m(\cos\theta)$, where P_m is the Legendre polynomial of order m and θ is the spherical polar angle. According to Sommerfeld's limit process, P_m and the Bessel function J_0 are related by

$$J_0(kr) = \lim_{m\to\infty} P_m\left(\sqrt{1-\frac{k^2r^2}{m^2}}\right).$$

A comparison of $J_0(kr)$ and $P_m\left(\sqrt{1-k^2r^2/m^2}\right)$ is shown in Fig. 20 for three increasing values of m. Further relations between Bessel functions and Legendre functions are discussed by G. N. Watson, *loc. cit.*, §5.71, pp. 155–157.

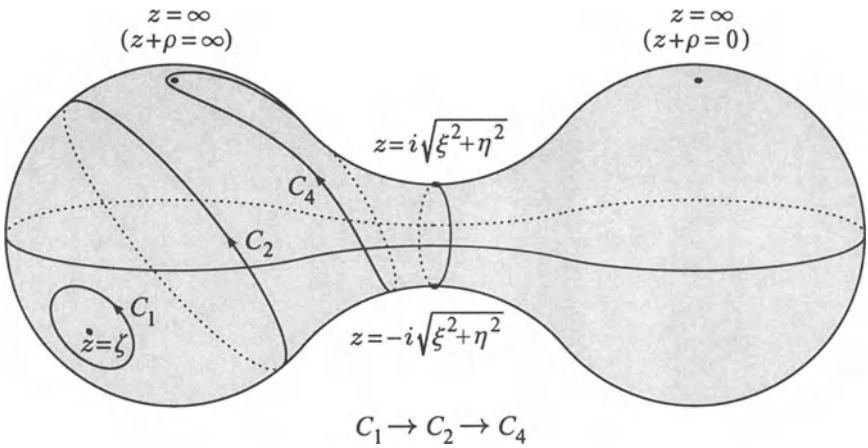


FIG. 21. Integration paths on a two-sheeted Riemann surface with branch points at $z = \pm i\sqrt{\xi^2 + \eta^2}$, $z + \rho = 0$, and $z + \rho = \infty$.

17. (page 336) The two-sheeted Riemann surface in Fig. 16(b) now has branch points at the points $z = \infty$. On the ρ^+ sheet, the point $z = \infty$ corresponds to $z + \rho = \infty$, while the point $z = \infty$ on the ρ^- sheet corresponds to $z + \rho = 0$. The original integration contour C_1 around $z = \zeta$ may be deformed into the contour C_4 shown in Fig. 21, but may not cross over the branch point at $z = \infty$. Under the change of variables $z = i\sqrt{\xi^2 + \eta^2}\cos\alpha$ discussed above, the contour C_4 is transformed into Sommerfeld's path (A) in the α plane.

18. (page 337) A factor of $\frac{1}{2\pi}$ is omitted from the left-hand side of the following equation.

19. (page 338) The two-sheeted Riemann surface in Fig. 21 is now cut from $z + \rho = \infty$ to $z + \rho = 0$, as shown in Fig. 22. In order to obtain an integration path that surrounds the cut, we may begin with the two paths C_1 and C_1'. The path C_1 may be deformed from C_1 to C_2 to C_5, and the path C_1' may be deformed from C_1' to C_2' to C_5'. If the integral along C_5 corresponds to Sommerfeld's equation (5) on page 332 with $f = \log \mathsf{Z}$, then the integral along C_5' corresponds to Sommerfeld's equation (6) on page 333 with $f = \log \mathsf{Z}$. The integrals over the portions of C_5 and C_5' that run on the circular neck of the surface in Fig. 22 cancel, leaving an integral along a path that surrounds the

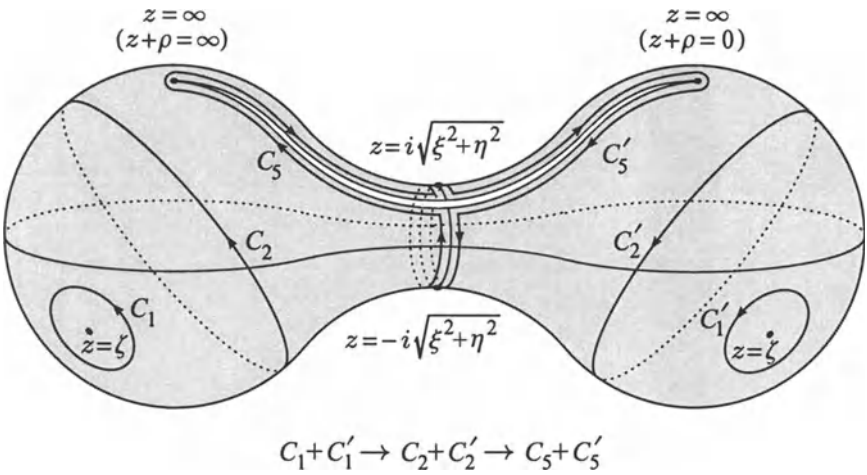


FIG. 22. Integration paths on a two-sheeted Riemann surface with a branch cut between $z+\rho=0$ and $z+\rho=\infty$.

branch cut. After the change of variables $z = i\sqrt{\xi^2+\eta^2}\cos\alpha$, the paths C_5 and C_5' correspond to the upper and lower rectangular paths in Sommerfeld's Fig. 3 on page 338.

If we denote the integration path that surrounds the branch cut in Fig. 22 by C_B, then

$$\int_{C_B} \log\left(\frac{\xi+i\eta}{z+\rho}\right)\frac{1}{(\zeta-z)^{m+1}}\frac{dz}{\rho} = \int_{C_B^+} \log\left(\frac{\xi+i\eta}{z+\rho}\right)\frac{1}{(\zeta-z)^{m+1}}\frac{dz}{\rho} + \\ + \int_{C_B^-} \log\left(\frac{\xi+i\eta}{z+\rho}\right)\frac{1}{(\zeta-z)^{m+1}}\frac{dz}{\rho},$$

where C_B^+ and C_B^- denote the paths on the two sides of the cut. The quantity ρ is continuous on the two-sheeted Riemann surface, but the logarithm differs on C_B^+ and C_B^- by $2\pi i$. Therefore

$$\left[\log\left(\frac{\xi+i\eta}{z+\rho}\right)\right]_{C_B^-} = \left[\log\left(\frac{\xi+i\eta}{z+\rho}\right)\right]_{C_B^+} - 2\pi i$$

$$\left[\frac{1}{(\zeta-z)^{m+1}\rho}\right]_{C_B^-} = \left[\frac{1}{(\zeta-z)^{m+1}\rho}\right]_{C_B^+}$$

$$[dz]_{C_B^-} = -[dz]_{C_B^+}$$

and

$$\int_{C_B} \log\left(\frac{\xi+i\eta}{z+\rho}\right) \frac{1}{(\zeta-z)^{m+1}} \frac{dz}{\rho} =$$

$$= \int_{C_B^+} \left\{ \log\left(\frac{\xi+i\eta}{z+\rho}\right) - \left[\log\left(\frac{\xi+i\eta}{z+\rho}\right) - 2\pi i\right] \right\} \frac{1}{(\zeta-z)^{m+1}} \frac{dz}{\rho}$$

$$= 2\pi i \int_{C_B^+} \frac{1}{(\zeta-z)^{m+1}} \frac{dz}{\rho}.$$

Under the change of variables $z = i\sqrt{\xi^2+\eta^2}\cos\alpha$, the contour C_B^+ is transformed into Sommerfeld's path (B) in the α plane.

20. (page 339) Sommerfeld's derivation on page 339 of the integral representation for the expression

$$e^{\frac{i\nu\pi}{2}} J_\nu(x) - e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x)$$

begins by subtracting equation (5b) on page 339 from equation (5a) on page 339, with the result

$$2\pi e^{\frac{i\nu\pi}{2}} J_\nu(x) - 2\pi e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x) =$$

$$= \int_A e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha - \int_A e^{ix\cos\alpha} e^{-i\nu\alpha}\, d\alpha, \tag{5}$$

where the integration path A is shown in Fig. 23. If the successive changes $\beta = -\alpha$, $\alpha = \beta$ of the integration variable are made in the second integral on the right-hand side of equation (5), then equation (5) becomes

$$2\pi e^{\frac{i\nu\pi}{2}} J_\nu(x) - 2\pi e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x) =$$

$$= \int_A e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha + \int_{A'} e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha$$

$$= \int_{B_1} e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha + \int_{B_2} e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha$$

$$= \int_{B_1} e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha - \int_{B_3} e^{ix\cos\alpha} e^{i\nu\alpha}\, d\alpha, \tag{6}$$

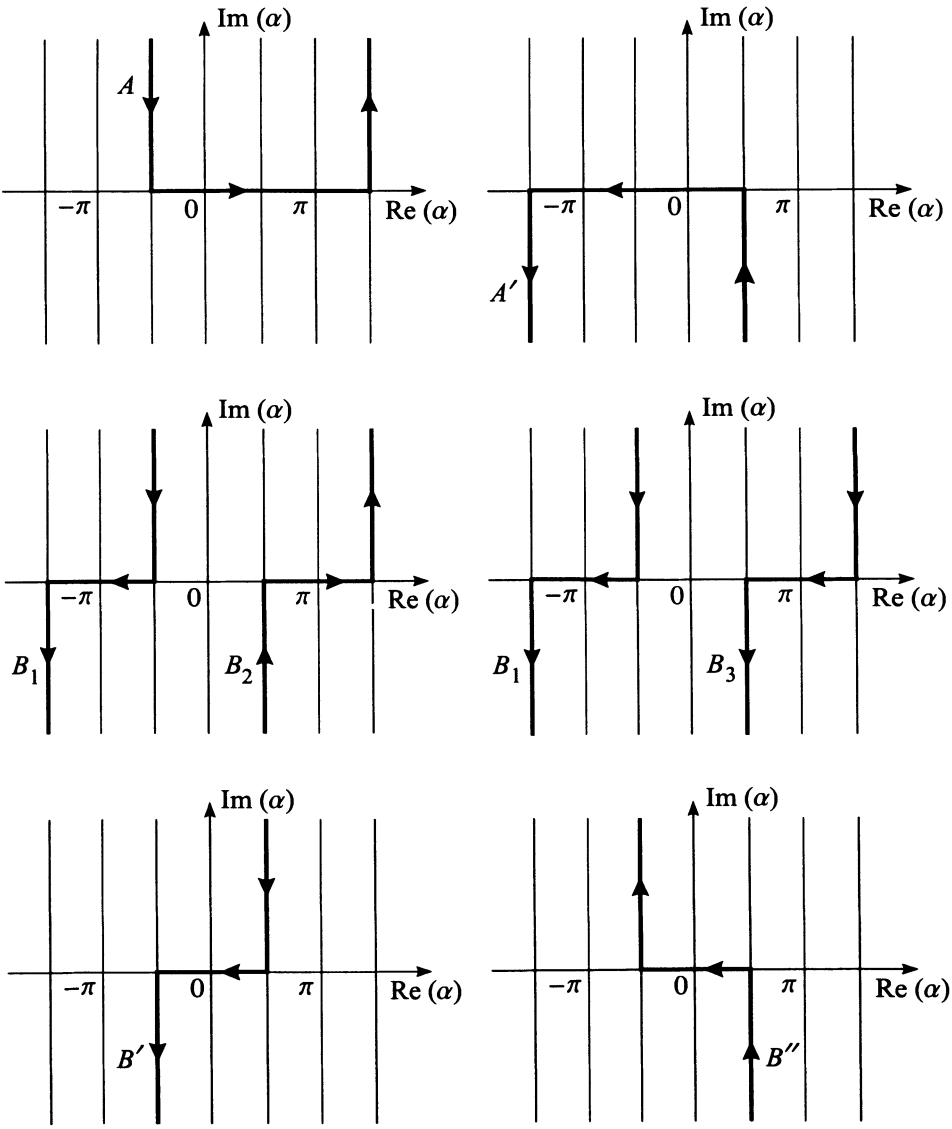


FIG. 23. Integration paths for the derivation of the relation among $U_\nu(x)$, $J_\nu(x)$, and $J_{-\nu}(x)$.

where the paths A', B_1, B_2, and B_3 are defined in Fig. 23. Making the change of variables $\gamma_1 = \alpha + \pi$ in the first integral on the right-hand side of equation (6) and the change of variables $\gamma_2 = \alpha - \pi$ in the second integral on the right-hand side of equation (6), equation (6) becomes

$$
\begin{aligned}
2\pi e^{\frac{i\nu\pi}{2}} J_\nu(x) - 2\pi e^{-\frac{i\nu\pi}{2}} J_{-\nu}(x) = \\
&= \int_{B'} e^{ix\cos(\gamma_1-\pi)} e^{i\nu(\gamma_1-\pi)}\, d\gamma_1 - \int_{B'} e^{ix\cos(\gamma_2+\pi)} e^{i\nu(\gamma_2+\pi)}\, d\gamma_2 \\
&= e^{-i\nu\pi} \int_{B'} e^{-ix\cos\gamma_1} e^{i\nu\gamma_1}\, d\gamma_1 - e^{i\nu\pi} \int_{B'} e^{-ix\cos\gamma_2} e^{i\nu\gamma_2}\, d\gamma_2 \\
&= (e^{-i\nu\pi} - e^{i\nu\pi}) \int_{B'} e^{-ix\cos\alpha} e^{i\nu\alpha}\, d\alpha, \qquad (7)
\end{aligned}
$$

where the path B' in Fig. 23 is equivalent to the path B' in Sommerfeld's Fig. 3.

21. (page 339) If $-i$ is exchanged with i in equation (7), the result is

$$\frac{1}{2i} \int_{B''} e^{ix\cos\alpha} e^{i\nu\left(-\alpha-\frac{\pi}{2}\right)}\, d\alpha = \pi \frac{J_{-\nu}(x) - e^{-i\nu\pi} J_\nu(x)}{2\sin\nu\pi},$$

where the path B'' is defined in Fig. 23. Making the successive changes $\beta = -\alpha$, $\alpha = \beta$ of the integration variable now gives

$$\frac{1}{2i} \int_{B} e^{ix\cos\alpha} e^{i\nu\left(\alpha-\frac{\pi}{2}\right)}\, d\alpha = -\pi \frac{J_{-\nu}(x) - e^{-i\nu\pi} J_\nu(x)}{2\sin\nu\pi}, \qquad (8)$$

where the path of integration B is equivalent to the path (B) in Sommerfeld's Fig. 3. Equation (8) is the correct form of Sommerfeld's equation (6) on page 339.

22. (page 340) The evaluation of the limit $\nu \to n$ of the right-hand side of Sommerfeld's equation (6) on page 339 and the properties of the resulting function are discussed by G. N. Watson, *loc. cit.*, §3.5, pp. 57–73.

23. (page 341) Sommerfeld has omitted a negative sign in the second of the following equalities.

24. (page 342) A specific integration path (A) in the α plane and its image in the $\cos\alpha$ plane are shown in Fig. 24.

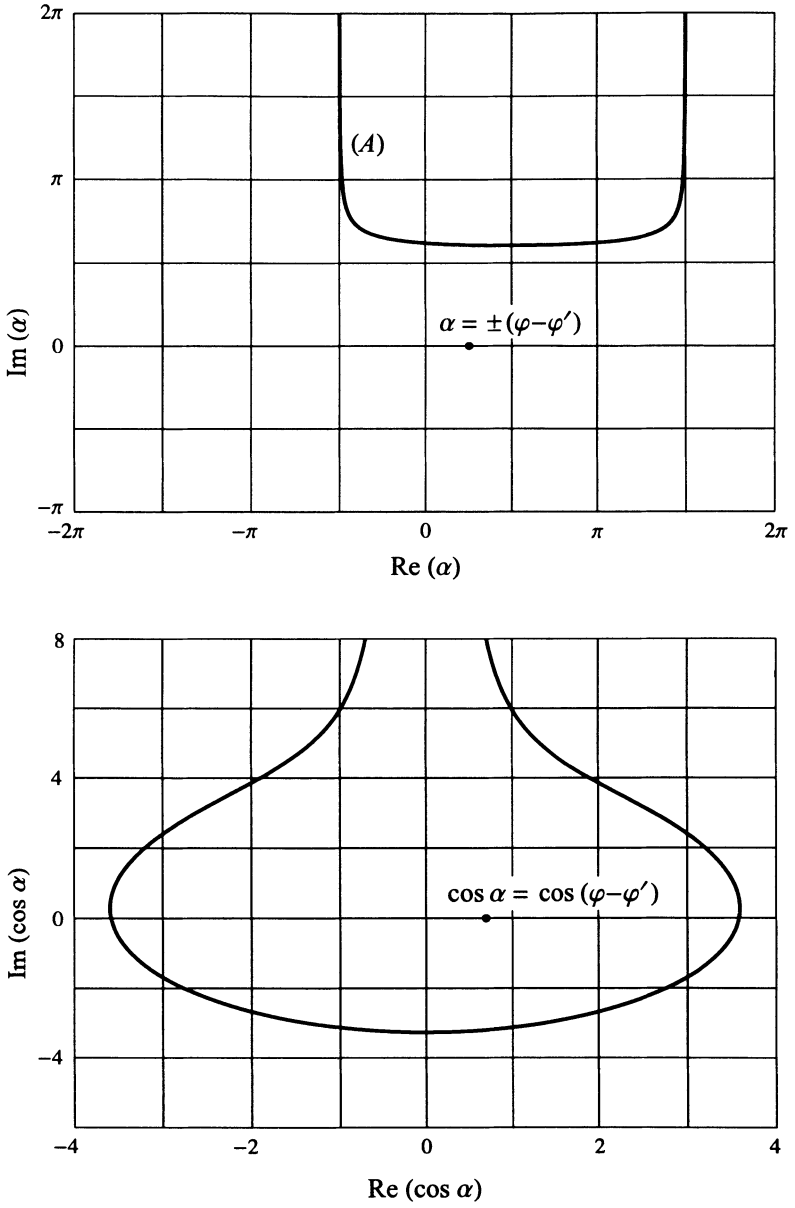


FIG. 24. Integration path (A) in the α plane and the corresponding path in the $\cos\alpha$ plane.

25. (page 343) A negative sign is again missing in the second of the following equalities.

26. (page 344) A modified form of this identity is given by I. S. Gradshteyn and I. M. Ryzhik in *Table of Integrals, Series and Products* (Academic Press, 5th ed., 1994, p. 41, 1.395(1)).

27. (page 345) The use of a Riemann surface in the solution of a boundary value problem involving a wedge-shaped body is discussed in note 37.

28. (page 347) If the function

$$f = \frac{1}{1 - \dfrac{\lg \mathsf{Z}}{\lg \mathsf{Z}'}}, \qquad \mathsf{Z}' = e^{i\varphi' - \frac{i\pi}{2}}$$

is substituted into equation (13) on page 334, the resulting function u' is

$$u' = -i\left(\varphi' - \frac{\pi}{2}\right)\frac{1}{2\pi i}\int \frac{2\alpha}{(\varphi - \varphi')^2 - \alpha^2}\, e^{ikr\cos\alpha}\, d\alpha.$$

The constant factor $-i\left(\varphi' - \dfrac{\pi}{2}\right)$ that Sommerfeld omits would seem to be important when $\varphi' = \dfrac{\pi}{2}$.

29. (page 347) The limit can be obtained, for example, by expanding the trigonometric functions in equation (5) on page 343 as $n \to \infty$ for fixed α.

30. (page 347) This identity follows from the identities

$$\cos x - \cos y = -2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y)$$

and

$$\sin x = x\prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right) = x\prod_{k=-\infty}^{\infty}{}'\left(1 - \frac{x}{k\pi}\right),$$

where the product $\prod'$ does not contain the factor corresponding to $k = 0$.

31. (page 347) Sommerfeld's surrealistic summation of infinitely many light rays on a Riemann surface of infinitely many sheets is based on the following reasoning. Sommerfeld has just shown that the function $u'(r, \varphi - \varphi')$,

which represents light with incident direction φ' on a Riemann surface with infinitely many sheets, has the property

$$\sum_{k=-\infty}^{\infty} u'(r, \varphi + 2kn\pi - \varphi') = u^{(n)}(r, \varphi - \varphi'), \tag{9}$$

where $u^{(n)}(r, \varphi - \varphi')$ is the function u given in equation (5) on page 343, with the superscript added to explicitly display the dependence on the parameter n. The function $u^{(n)}(r, \varphi - \varphi')$ represents light with incident direction φ' on a Riemann surface with n sheets, and is periodic in φ with period $2\pi n$. (In Sommerfeld's notation, the function $u^{(1)}$ is designated as u_0.) The sum on the left-hand side of equation (9) is interpreted in the first place as the sum of the values of the single function u' at the overlying points $(r, \varphi + 2kn\pi)$ of the Riemann surface. But this is also

$$\sum_{k=-\infty}^{\infty} u'(r, \varphi - (\varphi' - 2kn\pi)),$$

which can be interpreted as the sum of infinitely many functions u' with incident directions $\varphi' - 2kn\pi$, evaluated at the point (r, φ).

32. (page 349) The double integral (6) on page 349 is obtained by projecting Z from the North pole and Z′ from the South pole.

33. (page 351) Using the starting function

$$f = \frac{\frac{1}{n}}{1 - \left(\frac{Z}{Z'}\right)^{\frac{1}{n}}}$$

and projecting Z and Z′ from the South pole, the double integral analogous to Sommerfeld's (5) on page 349 becomes

$$\frac{1}{2\pi n} \int d\alpha \, e^{ikr\cos\alpha} \frac{1}{2i} \int d\alpha' \, e^{-ikr'\cos\alpha'} \frac{1}{1 - e^{i(\varphi - \varphi' + \alpha + \alpha')/n}}. \tag{10}$$

By projecting Z from the North pole and Z′ from the South pole, we obtain the double integral

$$-\frac{1}{2\pi n} \int d\alpha \, e^{ikr\cos\alpha} \frac{1}{2i} \int d\alpha' \, e^{-ikr'\cos\alpha'} \frac{1}{1 - e^{i(\varphi - \varphi' - \alpha + \alpha')/n}},$$

which, after making the change of variables $\beta = -\alpha'$ and reversing the direction of the β integration, becomes

$$-\frac{1}{2\pi n}\int d\alpha\, e^{ikr\cos\alpha}\,\frac{1}{2i}\int d\beta\, e^{-ikr'\cos\beta}\,\frac{1}{1-e^{i(\varphi-\varphi'-\alpha-\beta)/n}}. \qquad (11)$$

Changing the name of the integration variable β back to α', the sum of the double integrals (10) and (11) gives

$$\begin{aligned} U &= \frac{1}{2\pi i n}\int d\alpha\, e^{ikr\cos\alpha}\frac{1}{2i}\int d\alpha'\, e^{-ikr'\cos\alpha'}\,\frac{\sin(\alpha+\alpha')/n}{\cos(\alpha+\alpha')/n-\cos(\varphi-\varphi')/n} \\ &= \frac{1}{2\pi i n}\,\frac{1}{2i}\int d\alpha'\, e^{-ikr'\cos\alpha'}\int d\alpha\, e^{ikr\cos\alpha}\,\frac{\sin(\alpha+\alpha')/n}{\cos(\alpha+\alpha')/n-\cos(\varphi-\varphi')/n}. \end{aligned} \qquad (12)$$

In equation (12), we first let $\beta = \alpha+\alpha'$ in the α integral, and then let $\gamma = -\alpha'$ in the α' integral. With the appropriate changes in the paths of integration, equation (12) then becomes

$$\begin{aligned} U &= \frac{1}{2\pi i n}\,\frac{1}{2i}\int d\gamma\, e^{-ikr'\cos\gamma}\int d\beta\, e^{ikr\cos(\beta+\gamma)}\,\frac{\sin\beta/n}{\cos\beta/n-\cos(\varphi-\varphi')/n} \\ &= \frac{1}{2\pi i n}\int d\beta\,\frac{\sin\beta/n}{\cos\beta/n-\cos(\varphi-\varphi')/n}\,\frac{1}{2i}\int d\gamma\, e^{ikr\cos(\beta+\gamma)}e^{-ikr'\cos\gamma}. \end{aligned} \qquad (13)$$

By Sommerfeld's equation (7) on page 349 and equation (8) on page 350, equation (13) becomes

$$U = \frac{1}{2\pi i n}\int d\beta\,\frac{\sin\beta/n}{\cos\beta/n-\cos(\varphi-\varphi')/n}\,U_0(kR')$$

$$R'^2 = r^2 + r'^2 - 2rr'\cos\beta,$$

which is equivalent to Sommerfeld's equation (9) on page 351.

34. (page 351) The function $U_0(z)$ has a logarithmic singularity at $z = 0$, and is single-valued on a Riemann surface of infinitely many sheets. If the points of this surface are represented by the complex numbers $re^{i\varphi}$, then

$$U_0(re^{i(\varphi+2\pi)}) - U_0(re^{i\varphi}) = 2\pi i J_0(re^{i\varphi}).$$

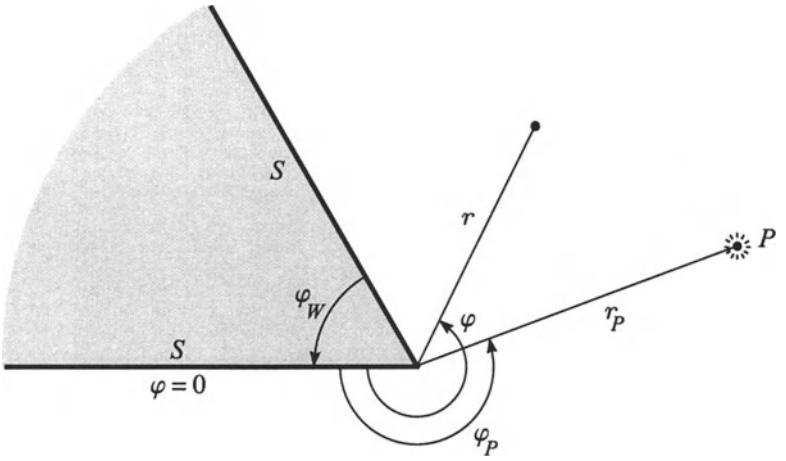


FIG. 25. Electrostatic point charge P in the presence of a perfectly conducting wedge.

A derivation of the analogous relations for the Hankel functions $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$ is given by Sommerfeld in his *Partial Differential Equations in Physics* (Academic Press, 1964, pp. 312–316).

35. (page 351) The asymptotic form that Sommerfeld gives for $U_0(z)$ shows that $U_0(kR)$ should be used as the point source solution for two-dimensional wave propagation problems with $e^{-i\omega t}$ time dependence.

36. (page 352) The "realizations of the real axis in the α plane" are the values of α corresponding to real values of R'. The equation near the top of page 352 that defines the quantity a contains a typographical error; a should be defined as

$$a = \operatorname{arc\,cosh} \frac{r^2 + r'^2}{2rr'}.$$

37. (page 354) Certain boundary value problems in a two-dimensional spatial domain that spans an angular sector $0 < \varphi < n\pi/m$ can be solved by reformulating the physical problem on a Riemann surface of n sheets. As an example, we consider the two-dimensional electrostatic problem shown in Fig. 25. A point charge P generates an electrostatic field in the presence of a perfectly conducting wedge with boundary surfaces S. The polar angle of the

point charge P is φ_P. The wedge angle is $\varphi_W = \pi/3$, and the angular domain of the space where the electrostatic field is to be determined is $0 < \varphi < 5\pi/3$. As in note 3, we introduce the complex potential

$$v = \phi + i\psi,$$

where ϕ is the real electrostatic potential and $\psi = \text{constant}$ is a line of electrostatic force. The complex potential v satisfies the Laplace equation

$$\Delta v = 0$$

with the conditions

$$\phi = 0 \text{ for } \varphi = 0 \text{ and } \varphi = 5\pi/3$$

and

$$\phi = \infty \text{ as } \log \frac{1}{R} \text{ at } P,$$

where R denotes the distance from P.

To solve for v, we extend the domain of the problem to the Riemann surface of five sheets shown schematically in Fig. 26, and first find the complex electrostatic potential $u(P)$ due to a point charge P on this surface. The function $u(P)$ can be obtained by considering a conformal map from the Riemann surface of five sheets in Fig. 26 to a smooth plane, just as the function $u(P)$ in note 3 is obtained by considering a conformal map from the Riemann surface of two sheets in Fig. 9(a) to the smooth plane in Fig. 9(b). If the points of the Riemann surface of Fig. 26 are represented by the complex numbers $z = re^{i\varphi}$ and the points of the smooth plane are represented by the complex numbers $w = \rho e^{i\vartheta}$, then the appropriate conformal map here is

$$w = z^{1/5}, \quad \text{or } \rho = \sqrt[5]{r} \text{ and } \vartheta = \varphi/5.$$

The function $u(P)$ for the complex electrostatic potential on the Riemann surface of five sheets is therefore

$$u(P) = \frac{1}{2\pi} \log(z^{1/5} - z_P^{1/5}) = \frac{1}{2\pi} \log(\sqrt[5]{r}\, e^{i\varphi/5} - \sqrt[5]{r_P}\, e^{i\varphi_P/5}), \qquad (14)$$

where (r_P, φ_P) are the polar coordinates of the point P.

The electrostatic potential v for the physical problem of Fig. 25 can now be constructed by forming a symmetric arrangement of image charges on the five-sheeted Riemann surface. Such an arrangement is shown in Fig. 26, and its

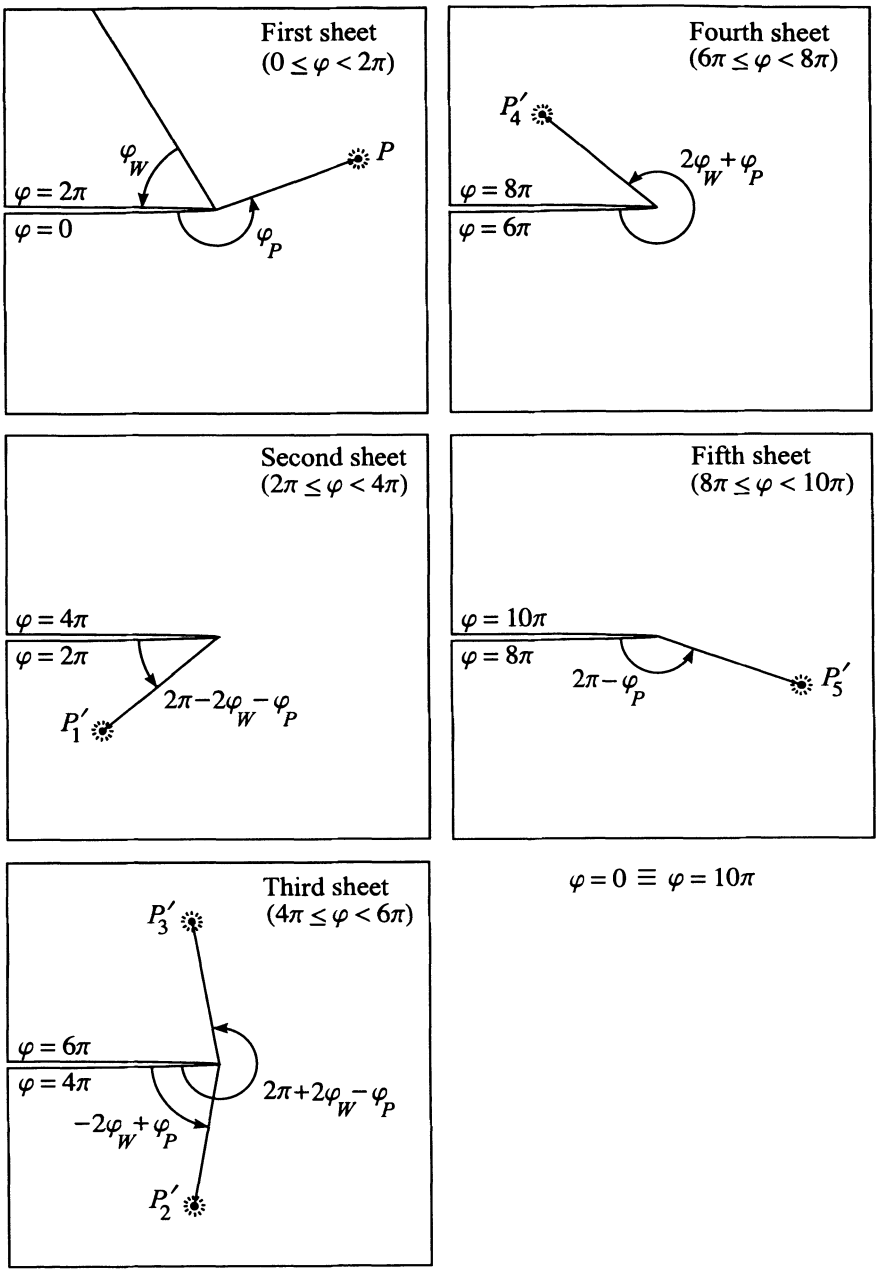


FIG. 26. Five-sheeted Riemann surface used in the solution of the electrostatic boundary value problem.

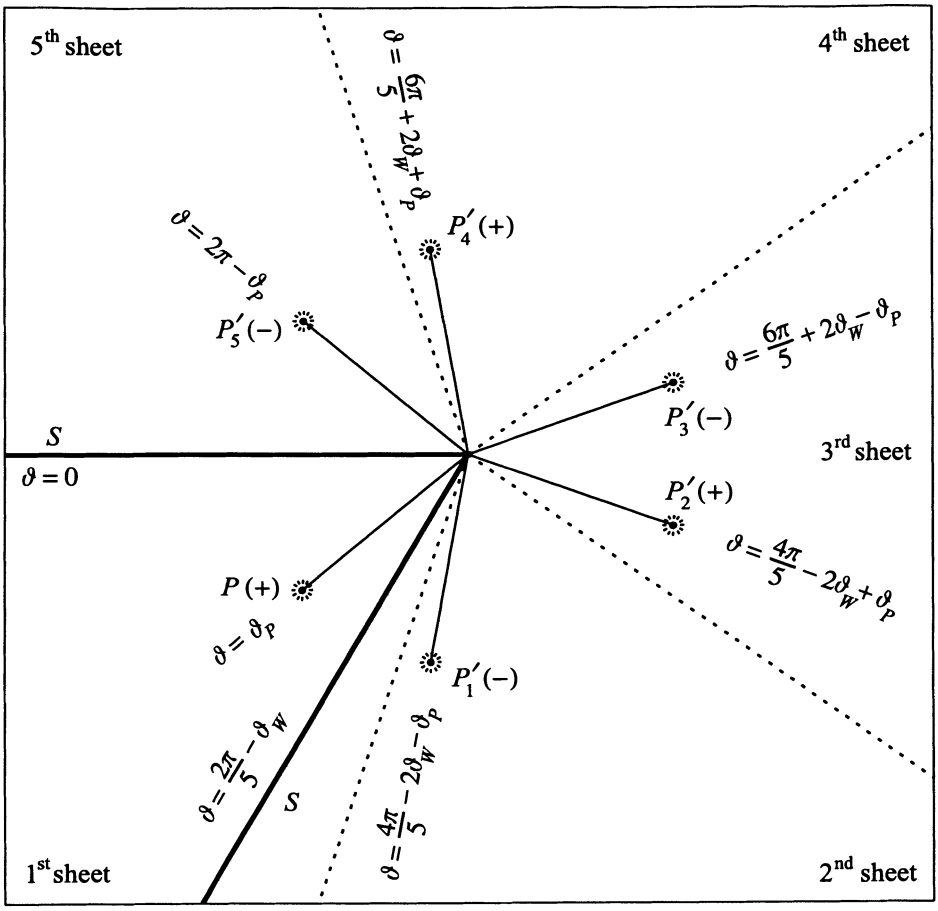


FIG. 27. Image charge distribution on the smooth w plane, with regions corresponding to the five sheets of the Riemann surface for z.

mapping onto the smooth w plane is shown in Fig. 27. The total complex potential of the set of charges in Fig. 26 is

$$v = u(P) - u(P_1') + u(P_2') - u(P_3') + u(P_4') - u(P_5'). \qquad (15)$$

Because of the symmetry of the distribution of the image charges, it is seen (most easily in Fig. 27) that the real part of v vanishes on the surfaces S of the wedge. The real part of v, evaluated in the physical domain $0 < \varphi < 5\pi/3$, thus gives the solution for the real electrostatic potential of the original physical

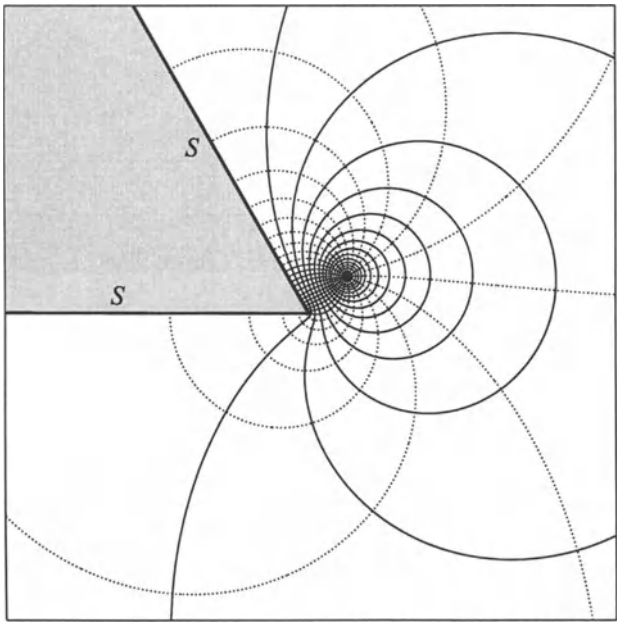


FIG. 28. Equipotentials (—) and lines of electrostatic force (···) for a point charge in the presence of a perfectly conducting wedge.

problem. Some equipotentials (lines of constant $\phi = \mathrm{Re}\,(v)$) and lines of electrostatic force (lines of constant $\psi = \mathrm{Im}\,(v)$) are shown in Fig. 28 for the case $\varphi_P = 5\pi/4$.

The function $u(P)$ defined by equation (14) is even more remarkable than the analogous function used in note 3. For example, the plane projection of an equipotential of the function $u(P)$ in equation (14) can have the form shown in Fig. 29. It is an interesting exercise to follow the curve in Fig. 29 as it winds around the five sheets of the Riemann surface of Fig. 26.

To solve the corresponding problem of two-dimensional monochromatic diffraction by the perfectly conducting wedge, it is necessary only to replace the function u in equations (14) and (15) by Sommerfeld's function U in equation (9) on page 351, evaluated for the special case $n = 5$.

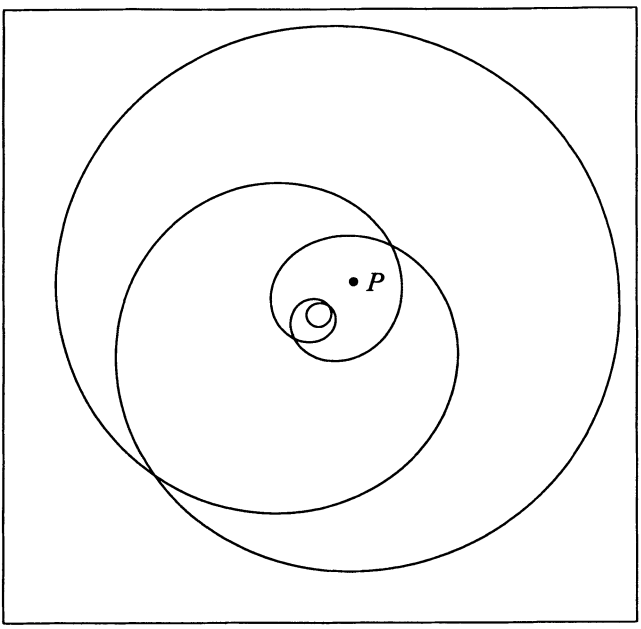


FIG. 29. Plane projection of an equipotential of the function $u(P)$.

Alternative and approximate methods for the solution of wedge diffraction problems are discussed by F. Oberhettinger in his review paper "On the Diffraction and Reflection of Waves and Pulses by Wedges and Corners," (*Journal of the National Bureau of Standards*, Vol. 61, No. 5, 1958, pp. 343–365).

38. (page 355) Because of the limit process definitions for the variables ξ', η', ζ', and α' that Sommerfeld gives on page 349, the integration over the α' variable cannot be taken directly over the path B that is defined in §4. If we use the starting function $f(\mathsf{Z}') = (\mathsf{Z}')^{-\nu}$ and then apply the limit process with projection from the South pole and the appropriate numerical factor, the integral

$$\frac{1}{2i}\int_{B'} d\alpha'\, e^{-ikr'\cos\alpha'}\, e^{-i\nu(\varphi'-\alpha'-\frac{\pi}{2})}$$

is obtained. Since the integrand must vanish at the endpoints of the path of integration, the path B' may be taken, for example, over the line segments that

connect the points

$$-\frac{3\pi}{2}+i\infty,\quad -\frac{3\pi}{2},\quad -\frac{\pi}{2},\quad -\frac{\pi}{2}-i\infty.$$

Making the change of variables $\beta=\alpha'+\pi$, we obtain

$$\frac{1}{2i}\int_{B'} d\alpha'\, e^{-ikr'\cos\alpha'}\, e^{-i\nu(\varphi'-\alpha'-\frac{\pi}{2})} = e^{-i\nu\varphi'}\frac{1}{2i}\int_{B} d\beta\, e^{ikr'\cos\beta}\, e^{i\nu(\beta-\frac{\pi}{2})}$$

$$= e^{-i\nu\varphi'}U_\nu(kr').$$

Similarly, using the starting function $f(\mathsf{Z}')=(\mathsf{Z}')^\nu$ with projection from the North pole gives the integral

$$-\frac{1}{2i}\int_{B'} d\alpha'\, e^{-ikr'\cos\alpha'}\, e^{i\nu(\varphi'+\alpha'-\frac{\pi}{2})}.$$

Again making the change of variables $\beta=\alpha'+\pi$, we obtain

$$-\frac{1}{2i}\int_{B'} d\alpha'\, e^{-ikr'\cos\alpha'}\, e^{i\nu(\varphi'+\alpha'-\frac{\pi}{2})} =$$

$$= -e^{i\nu\varphi'}e^{-i\nu\pi}\frac{1}{2i}\int_{B} d\beta\, e^{ikr'\cos\beta}e^{i\nu(\beta-\frac{\pi}{2})} = -e^{i\nu\varphi'}e^{-i\nu\pi}U_\nu(kr').$$

Thus, Sommerfeld's summary at the end of §4 should be augmented by the following:

From the limit process there results

	on the path B'
from $(\mathsf{Z}')^{-\nu}$	$e^{-i\nu\varphi'}U_\nu(kr')$

by projection from the South pole; and there results

from $(\mathsf{Z}')^{\nu}$	$-e^{-i\nu\pi}e^{i\nu\varphi'}U_\nu(kr')$

by projection from the North pole of the unit sphere.

If Sommerfeld's equation for $f(\mathsf{Z})+f(\mathsf{Z}')$ on page 355 is integrated on the path A with respect to the variable α and on the path B' with respect to the

variable α', then the augmented summary can be used to obtain equation (12) on page 355.

39. (page 358) A specific integration path (A) in the α plane and its image in the $\cos\frac{\alpha}{2}$ plane are shown in Fig. 30.

40. (page 360) The D-line is a spectrographic line corresponding to the yellow light emitted by a sodium lamp.

41. (page 361) Wolfgang Pauli (1900–1958), a student of Sommerfeld who won the 1945 Nobel Prize in physics for the formulation of the exclusion principle in quantum mechanics, derived an asymptotic ($kr \to \infty$) approximation of the function u that is also valid inside the transition region ("On Asymptotic Series for Functions in the Theory of Diffraction of Light," *Physical Review*, Vol. 54, 1938, pp. 924–931). Pauli cites an earlier attempt by Sommerfeld himself ("Asymptotische Darstellung von Formeln aus der Beugungstheorie des Lichtes," *Journal für die reine und angewandte Mathematik*, Vol. 158, 1927/28, pp. 199–208) to derive such an approximation, but concludes that Sommerfeld's results "were not satisfactory, because the neglected terms were not sufficiently small in comparison to the retained terms."

42. (page 362) The plate (*Tafel*) appears on page 67.

43. (page 364) Sommerfeld's *Tafel*, which is unfortunately omitted from his *Gesammelte Schriften*, is presented in its original form on page 67. The plate is a remarkably accurate plot of the nodal lines of the real part of the function u defined by equation (5) on page 359. To emphasize this point, we show in Fig. 31 a computational reproduction of the plate, derived from a direct numerical evaluation of equation (5), without the use of Sommerfeld's asymptotic approximations. Fig. 31 is based on the proportions defined by the values of λ and e given by Sommerfeld in the footnote on page 360. (In the original *Tafel*, the distance between the vertical lines on the first sheet is exactly $\lambda/2 = 5$ mm.) The large amplitude oscillations of the nodal lines outside the parabolic transition strip in Fig. 1 of the *Tafel* are probably an intentional exaggeration. Since Fig. 31 is based directly on a numerical evaluation of equation (5) on page 359, it is valid both inside and outside the transition strip. It is seen from Fig. 31 that the character of the function u on the first sheet does not change qualitatively as the parabola that bounds the transition strip is crossed.

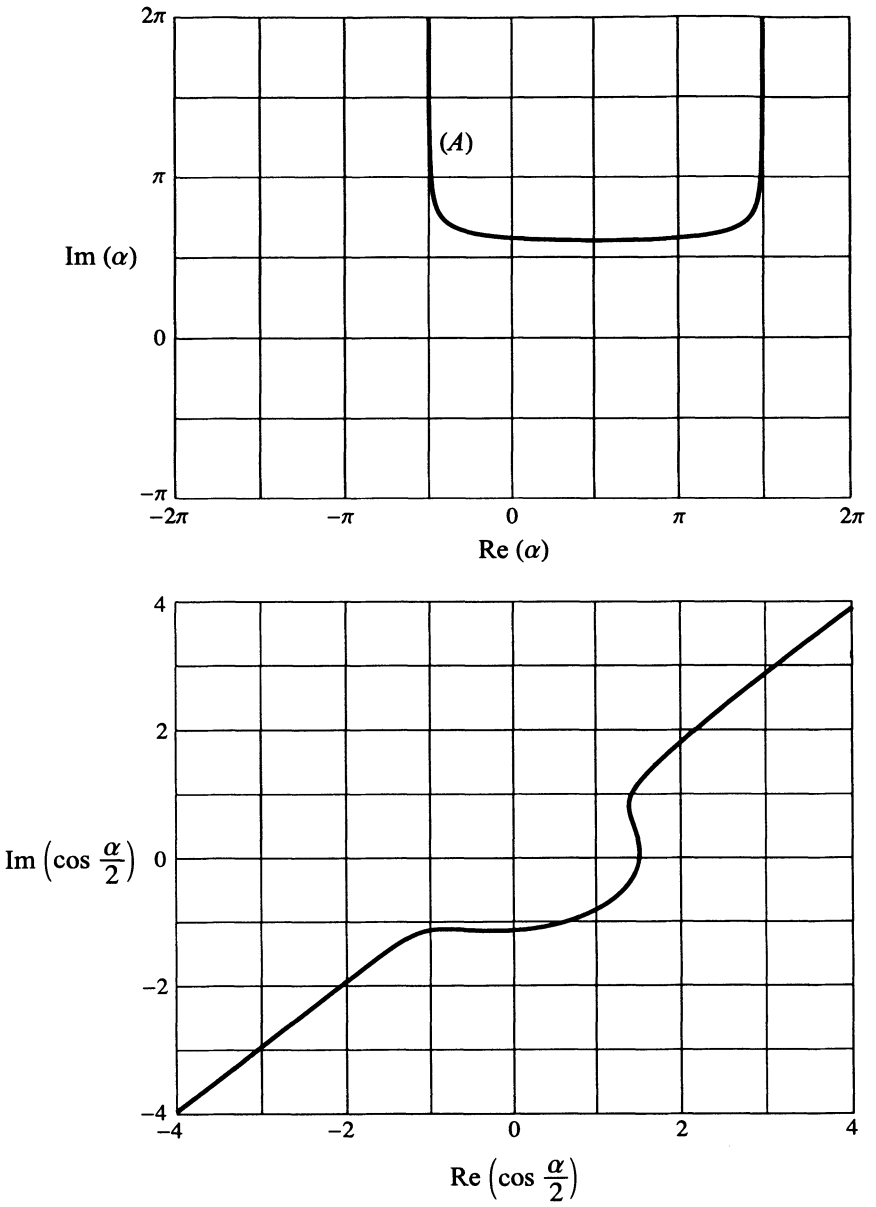


FIG. 30. Integration path (A) in the α plane and the corresponding path in the $\cos\frac{\alpha}{2}$ plane.

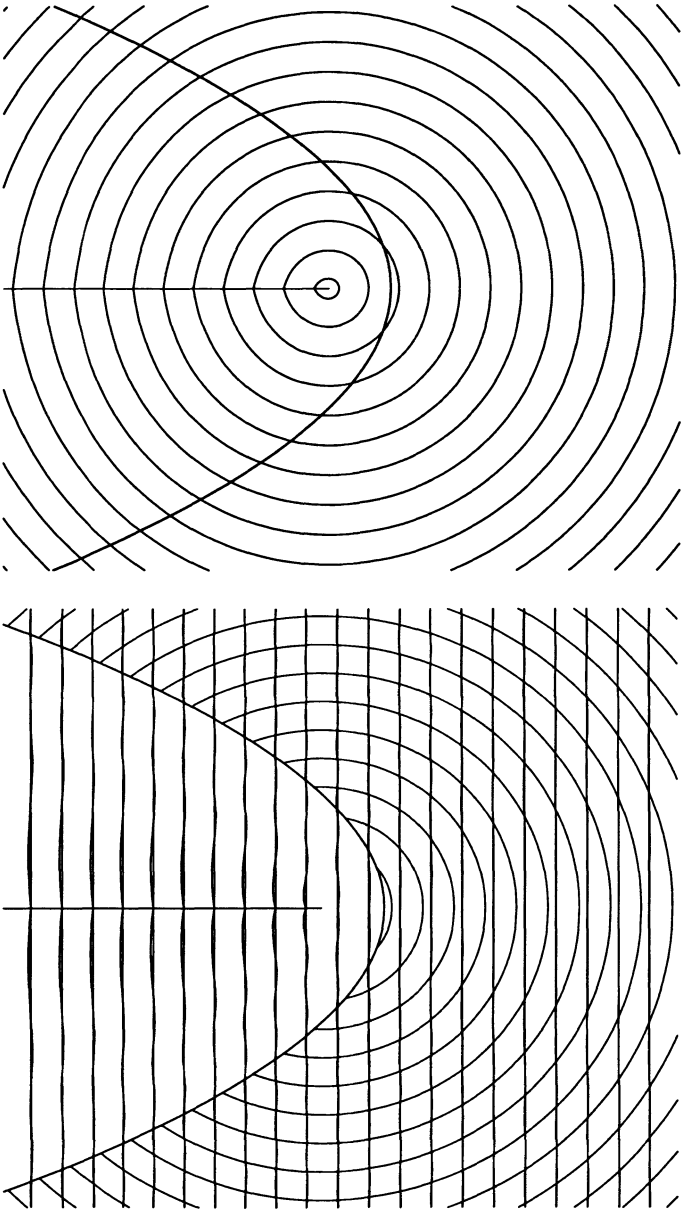


FIG. 31. Computational reproduction of Sommerfeld's *Tafel*, showing the nodal lines of Re (u) on the second sheet (upper half of the figure) and first sheet (lower half of the figure) of the Riemann surface.

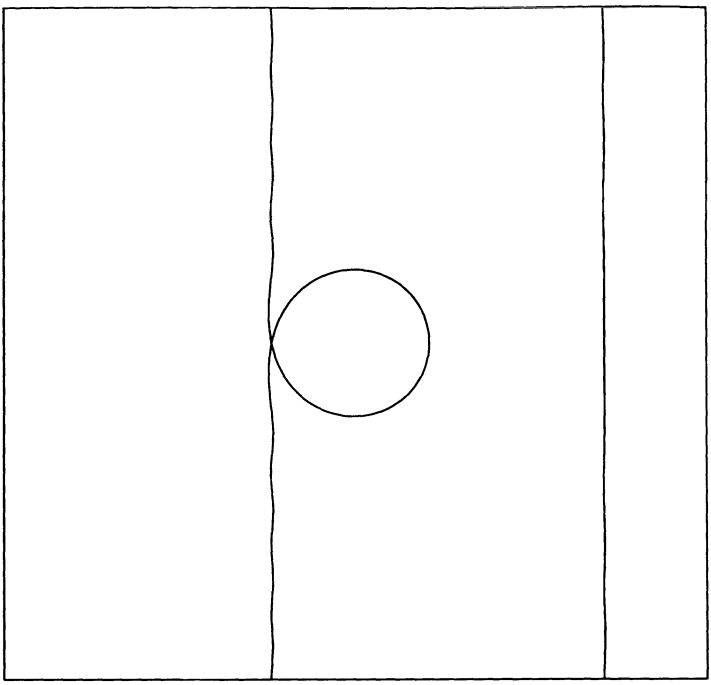


FIG. 32. Null curves of the function Re (u).

A simple continuation of the oscillating nodal lines into the transition strip gives an excellent depiction of u on the entire first sheet. Sommerfeld is careful not to make such an unjustified continuation, and does not attempt an accurate representation inside the transition strip.

Since the function u on page 359 depends on $\psi = \varphi - \varphi'$, the nodal lines of Re (u) for nonzero angles of incidence φ' can be obtained by a simple rotation.

44. (page 364) Two complete paths of Sommerfeld's null curves are shown in Fig. 32. The right-hand curve in Fig. 32 is take directly from first sheet in Fig. 31. The left-hand curve in Fig. 32 is a curve that intersects the axis of the parabola on the first sheet in Fig. 31, loops around the origin on the second sheet in Fig. 31, and then rejoins itself on the first sheet. The left-hand curve in Fig. 32 is analogous to the equipotential in Fig. 13(b) with the source P removed to infinity, so that the outer loop in Fig. 13(b) does not close. The equipotential in Fig. 13(b), however, is static, while the null curves in Fig. 32 propagate with time as described by Sommerfeld on page 365.

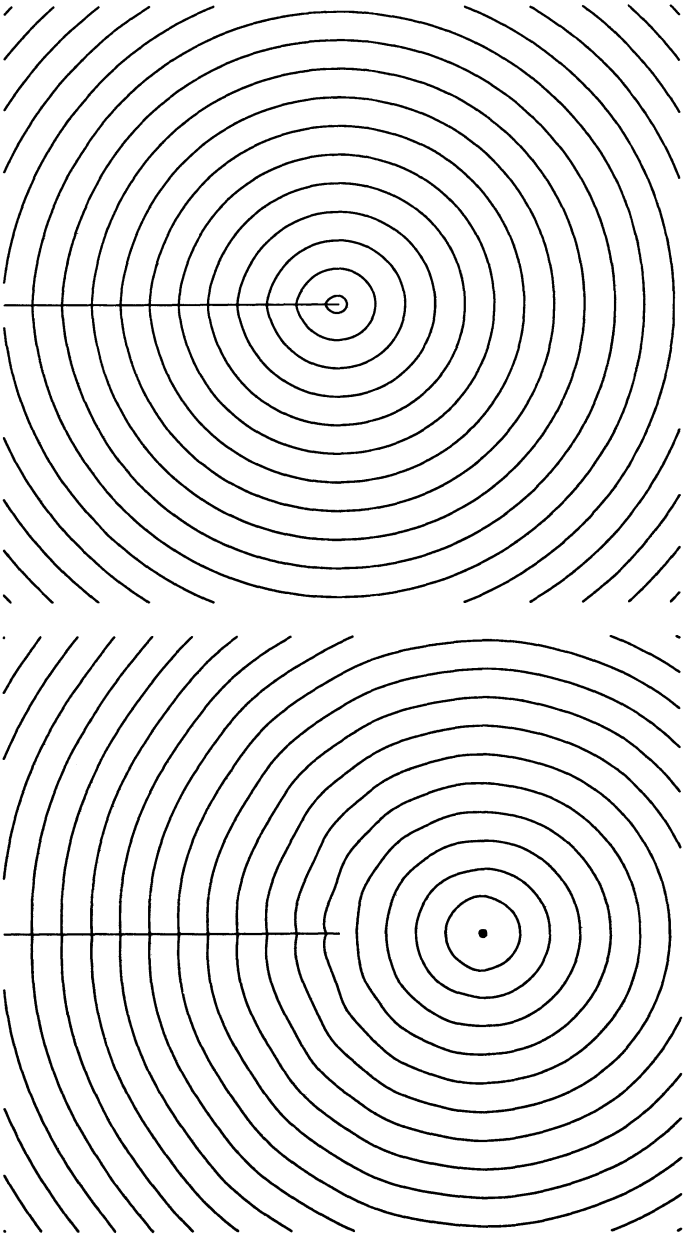


FIG. 33. Nodal lines of Re (U) on the second sheet (upper half of the figure) and first sheet (lower half of the figure) of the Riemann surface.

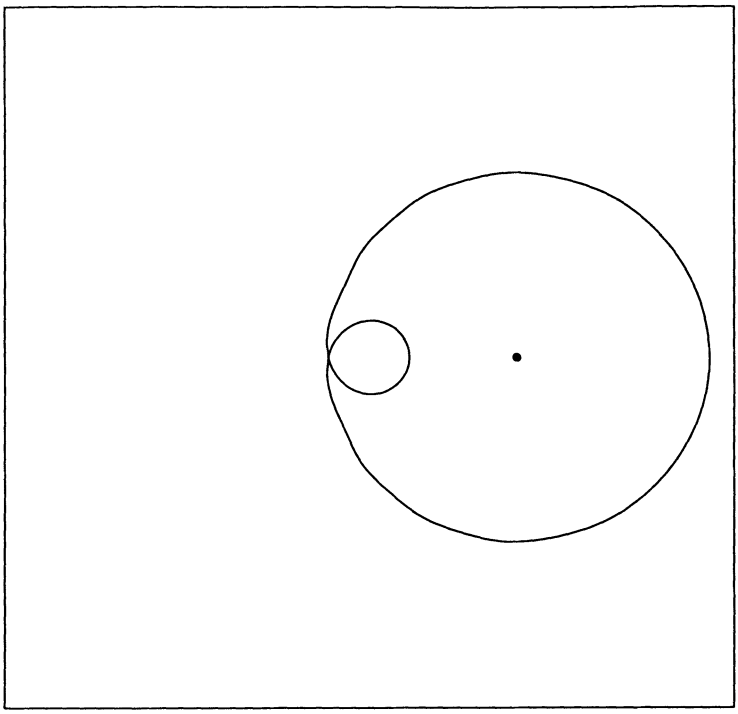


FIG. 34. Null curve of the function Re (U).

It is also possible to compute the nodal lines of the propagating wave field on the Riemann surface due to a point source in the finite region. These are the null curves of Sommerfeld's function U given by equation (9) on page 351. The null curves of U for the case $n = 2$ are shown in Fig. 33. The source is indicated by the solid dot on the first sheet of Fig. 33. Fig. 33 is based on a direct numerical evaluation of the contour integral in equation (9). Care is necessary to ensure that the branch structure of the integrand is consistent with Sommerfeld's discussion of his Fig. 4 on page 352. Some alternative forms of Sommerfeld's integral (9) are discussed by P. C. Clemmow ("A Note on the Diffraction of a Cylindrical Wave by a Perfectly Conducting Half-Plane," *Quarterly Journal of Mechanics and Applied Mathematics*, Vol. III, Pt. 3, 1950, pp. 377–384.)

The complete path of a null curve in Fig. 33 that passes onto the second sheet has the form shown in Fig. 34. The analogy between the propagating null curve and the static equipotential of Fig. 13(b) is now even closer.

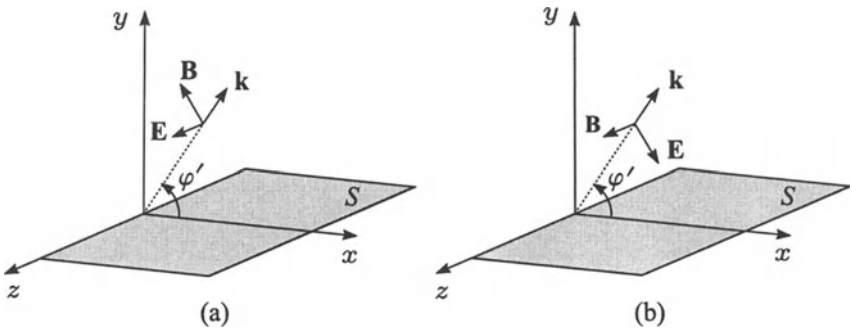


FIG. 35. Geometric configuration for Sommerfeld's half plane diffraction problem, with the polarization of the incident electromagnetic plane wave in case (a) and case (b).

45. (page 366) The geometric configuration of Sommerfeld's half plane diffraction problem is illustrated in Fig. 35. "Parallel light in a plane perpendicular to the edge of the screen" means an electromagnetic plane wave with a wave vector $\mathbf{k}$ in the x, y plane. For such a plane wave, the two independent polarizations can be taken as (a) the electric field vector $\mathbf{E}$ in the z-direction and the magnetic field vector $\mathbf{B}$ in the x, y plane; and (b) the magnetic field vector $\mathbf{B}$ in the z-direction and the electric field vector $\mathbf{E}$ in the x, y plane. The two cases are illustrated in Fig. 35. For the $e^{i(\mathbf{k}\cdot\mathbf{x}+\omega t)}$ space-time dependence used by Sommerfeld, the plane wave propagates towards the screen in the direction of $-\mathbf{k}$. Sommerfeld assumes that the total electromagnetic field in the presence of the screen can be decomposed into the same two independent polarizations (a) and (b).

In case (a), Maxwell's equations reduce as follows:

$$\nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \qquad \frac{\partial E_z}{\partial z} = 0 \tag{a.1}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \qquad \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{a.2}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \quad \Rightarrow \quad \begin{cases} -\dfrac{\partial B_y}{\partial z} = 0 & \text{(a.3)} \\ \dfrac{\partial B_x}{\partial z} = 0 & \text{(a.4)} \\ \dfrac{\partial B_y}{\partial x} - \dfrac{\partial B_x}{\partial y} - \dfrac{1}{c}\dfrac{\partial E_z}{\partial t} = 0 & \text{(a.5)} \end{cases}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{\partial E_z}{\partial y} + \dfrac{1}{c}\dfrac{\partial B_x}{\partial t} = 0 & \text{(a.6)} \\ -\dfrac{\partial E_z}{\partial x} + \dfrac{1}{c}\dfrac{\partial B_y}{\partial t} = 0 & \text{(a.7)} \\ 0 = 0. & \text{(a.8)} \end{cases}$$

Equations (a.1), (a.3), and (a.4) show that the field is two-dimensional, and equations (a.5)–(a.7) imply the two-dimensional wave equation

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 E_z}{\partial t^2} = 0$$

for the electric field component E_z.

In case (b), Maxwell's equations reduce in an analogous way:

$$\nabla \cdot \mathbf{E} = 0 \quad \Rightarrow \qquad \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = 0 \qquad \text{(b.1)}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \qquad \frac{\partial B_z}{\partial z} = 0 \qquad \text{(b.2)}$$

$$\nabla \times \mathbf{B} - \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{\partial B_z}{\partial y} - \dfrac{1}{c}\dfrac{\partial E_x}{\partial t} = 0 & \text{(b.3)} \\ -\dfrac{\partial B_z}{\partial x} - \dfrac{1}{c}\dfrac{\partial E_y}{\partial t} = 0 & \text{(b.4)} \\ 0 = 0 & \text{(b.5)} \end{cases}$$

$$\nabla \times \mathbf{E} + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = 0 \quad \Rightarrow \quad \begin{cases} -\dfrac{\partial E_y}{\partial z} = 0 & \text{(b.6)} \\ \dfrac{\partial E_x}{\partial z} = 0 & \text{(b.7)} \\ \dfrac{\partial E_y}{\partial x} - \dfrac{\partial E_x}{\partial y} + \dfrac{1}{c}\dfrac{\partial B_z}{\partial t} = 0. & \text{(b.8)} \end{cases}$$

Equations (b.2), (b.6), and (b.7) show that the field is again two-dimensional, and equations (b.3), (b.4), and (b.8) imply the two-dimensional wave equation

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 B_z}{\partial t^2} = 0$$

for the magnetic field component B_z.

Sommerfeld writes X, Y, and Z for E_x, E_y, and E_z in case (a), and writes X, Y, and Z for B_x, B_y, and B_z in case (b).

The two polarizations (a) and (b) can also be obtained by first assuming that the $\mathbf{E}$ and $\mathbf{B}$ fields are independent of the coordinate z. This is the approach taken by B. B. Baker and E. T. Copson in *The Mathematical Theory of Huygens' Principle* (Chelsea Publishing Co., 3rd ed., 1987, pp. 130–132) and by M. Born and E. Wolf in *Principles of Optics* (Pergamon Press, 1964, pp. 560–561).

46. (page 366) When a conducting material is in the presence of an external time-harmonic electromagnetic field, the penetration depth of the field into the conductor depends in a complicated way on the properties of the conductor and the frequency of the field. If the conductivity σ of the conductor, the permittivity ϵ of the conductor, and the frequency ω of the field satisfy the condition

$$\left(\frac{\sigma}{\epsilon\omega}\right)^2 \gg 1,$$

the penetration depth δ is of the order

$$\delta \sim \sqrt{\frac{2}{\omega\mu\sigma}},$$

where μ is the permeability of the conductor (see, for example, J. A. Stratton, *Electromagnetic Theory*, McGraw-Hill, 1941, pp. 500–504). In the idealized limit of a perfect conductor, the conductivity σ is infinite, so that the penetration depth δ is zero and the field inside the conductor vanishes.

At an interface between two different media, Maxwell's equations imply that the tangential component of the electric field must be continuous across the interface (*ibid.*, pp. 34–38). This implies that the boundary condition for an electromagnetic field in the presence of a perfect conductor is that the tangential component of the electric field **E** must vanish at the surface of the conductor. In Sommerfeld's polarization case (a), this immediately gives the boundary condition

$$E_z = Z = 0 \text{ on } S.$$

In case (b), the boundary condition is that $E_x = 0$ on S, which by equation (b.3) implies that

$$\frac{\partial B_z}{\partial y} = \frac{\partial Z}{\partial n} = 0 \text{ on } S.$$

Sommerfeld's condition that Z should be finite everywhere has been broadened by Meixner and others to a more general "edge condition"; this edge condition is important in establishing the uniqueness of the solution of the diffraction boundary value problem (see, for example, Born and Wolf, *loc. cit.*, pp. 591–592).

47. (page 367) In a very helpful review of this translation, Sir Michael Berry has written that "Sommerfeld's paper marked the climax of humanity's struggles, over several millenia, to understand shadows."

48. (page 371) Jules Henri Poincaré (1854–1912) wrote extensively on the mathematical theory of optical phenomena. The first volume of his *Théorie mathématique de la lumière* was published in 1889, and a second volume followed in 1892. The paper to which Sommerfeld refers is the first part of a long two-part memoir motivated by experiments of the French physicist Louis-George Gouy (1854–1926). In this memoir, Poincaré considers the following two-dimensional diffraction problem. A beam of light is focused by a cylindrical lens on the edge of a perfectly conducting screen S, as shown in Fig. 36. Two different possibilities for the polarization of the incident beam are considered. In the case discussed here, the only nonzero component of the electric field vector is the component Z perpendicular to the plane of Fig. 36. (This is Sommerfeld's case (a) discussed in note 45.) The light is assumed to be monochromatic with radian frequency ω, and Z is assumed to vanish on the surface of the screen. It desired to find the total electric field generated by the interaction of the incident beam and the perfectly conducting screen.

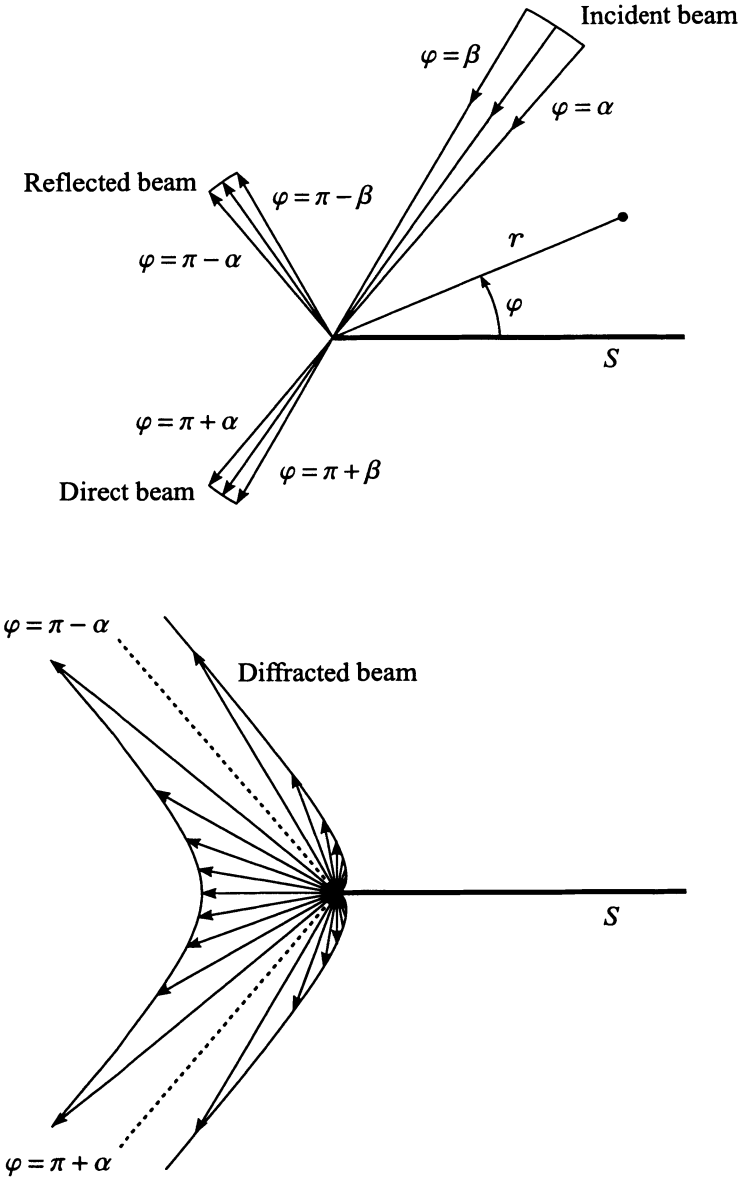


FIG. 36. Geometric configuration for Poincaré's two-dimensional diffraction analysis, with an incident cylindrical beam focused on the edge of the screen S.

Poincaré's analysis and results are fascinating both in themselves and in their relation to Sommerfeld's work. In order to help clarify Sommerfeld's comments about Poincaré's results, we summarize his analysis here with minor changes in the original notation and procedure. We are very grateful to Dr. Marc Herant for his translation of Poincaré's paper, and for many helpful technical discussions.

Poincaré begins by writing the total field Z as

$$Z = \mathrm{Re}\left(\mathsf{Z}e^{i\omega t}\right), \tag{16}$$

where

$$\mathsf{Z} = \sum_{n=0}^{\infty} a_n J_{\frac{n}{2}}(kr) \sin\frac{n\varphi}{2} \tag{17}$$

and the polar coordinates r, φ are defined in Fig. 36. The series (17) is a special case of Sommerfeld's series (2) on page 322, which Sommerfeld concludes on page 325 is a general representation for every continuous solution of the two-dimensional Helmholtz equation with a two-fold winding point at the origin. The $\cos\frac{n\varphi}{2}$ terms in Sommerfeld's general series (2) are omitted here in order to satisfy the boundary conditions $Z = 0$ at $\varphi = 0$ and $\varphi = 2\pi$. By assuming that Z is periodic in φ with period 4π, Poincaré anticipates Sommerfeld's use of a two-sheeted Riemann surface for the solution of the half plane diffraction problem.

Poincaré decomposes the total field Z as

$$Z = Z_i + Z_s, \tag{18}$$

where Z_i is the incident field that converges at the edge of screen and Z_s is a scattered field that diverges from the edge. Using the $e^{i\omega t}$ time dependence assumed in equation (16), the converging incident field is

$$Z_i = \mathrm{Re}\left(\mathsf{Z}_i e^{i\omega t}\right),$$

where

$$\mathsf{Z}_i = \sum_{n=0}^{\infty} b_n H^{(1)}_{\frac{n}{2}}(kr) \sin\frac{n\varphi}{2}$$

and the diverging scattered field is

$$Z_s = \mathrm{Re}\left(\mathsf{Z}_s e^{i\omega t}\right),$$

where

$$\mathsf{Z}_s = \sum_{n=0}^{\infty} c_n H_{\frac{n}{2}}^{(2)}(kr) \sin\frac{n\varphi}{2}.$$

Poincaré bases his analysis on the far-field ($kr \to \infty$) asymptotic approximations

$$\begin{aligned}\mathsf{Z} &\sim \sum_{n=0}^{\infty} a_n \sqrt{\frac{2}{\pi kr}}\, \cos(kr - \frac{\pi}{4} - \frac{n\pi}{4}) \sin\frac{n\varphi}{2} \\ &\sim \sum_{n=0}^{\infty} a_n \sqrt{\frac{2}{\pi kr}}\, \frac{e^{i(kr-\frac{\pi}{4}-\frac{n\pi}{4})}}{2} \sin\frac{n\varphi}{2} + \\ &\quad + \sum_{n=0}^{\infty} a_n \sqrt{\frac{2}{\pi kr}}\, \frac{e^{-i(kr-\frac{\pi}{4}-\frac{n\pi}{4})}}{2} \sin\frac{n\varphi}{2} \qquad (19)\end{aligned}$$

$$\mathsf{Z}_i \sim \sum_{n=0}^{\infty} b_n \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\frac{\pi}{4}-\frac{n\pi}{4})} \sin\frac{n\varphi}{2} \qquad (20)$$

$$\mathsf{Z}_s \sim \sum_{n=0}^{\infty} c_n \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr-\frac{\pi}{4}-\frac{n\pi}{4})} \sin\frac{n\varphi}{2}, \qquad (21)$$

which are obtained by using the far-field ($kr \to \infty$) asymptotic approximations of the functions $J_{\frac{n}{2}}(kr)$, $H_{\frac{n}{2}}^{(1)}(kr)$, and $H_{\frac{n}{2}}^{(2)}(kr)$.

Using equation (18) and matching the coefficients in the asymptotic approximations in equations (19), (20), and (21), it follows that

$$\frac{a_n}{2} = b_n, \quad \frac{a_n}{2} = c_n, \qquad (22)$$

so that the coefficients c_n of the scattered field are given in terms of the coefficients b_n of the incident field by

$$c_n = b_n. \qquad (23)$$

The coefficients c_n of the scattered field provide a formal solution to the diffraction problem, but Poincaré goes much further, and gives explicit results for the far-field angular dependence of the scattered field in terms of the far-field angular dependence of the incident field. The incident field Z_i in equation (20) is written as

$$\mathsf{Z}_i \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\frac{\pi}{4})} F(\varphi), \qquad (24)$$

where

$$F(\varphi) = \sum_{n=0}^{\infty} b_n \, e^{-\frac{in\pi}{4}} \sin\frac{n\varphi}{2}, \tag{25}$$

and the scattered Z_s field in equation (21) is written as

$$\mathsf{Z}_s \sim \sqrt{\frac{2}{\pi kr}} \, e^{-i(kr-\frac{\pi}{4})} G(\varphi), \tag{26}$$

where

$$G(\varphi) = \sum_{n=0}^{\infty} c_n \, e^{\frac{in\pi}{4}} \sin\frac{n\varphi}{2}. \tag{27}$$

From equation (25) it follows that

$$b_n = \frac{1}{\pi} \int_0^{2\pi} F(\varphi) \, e^{\frac{in\pi}{4}} \sin\frac{n\varphi}{2} \, d\varphi. \tag{28}$$

Using equations (23) and (28), the function $G(\varphi)$ in equation (27) that describes the far-field angular dependence of the scattered field is therefore given by

$$G(\varphi) = \sum_{n=0}^{\infty} \frac{1}{\pi} \int_0^{2\pi} F(\phi) \, e^{\frac{in\pi}{2}} \sin\frac{n\phi}{2} \, d\phi \, \sin\frac{n\varphi}{2}. \tag{29}$$

The series in equation (29) is now separated into a series for n even and a series for n odd, and the function $G(\varphi)$ is written as

$$G(\varphi) = \psi_1(\varphi) + \psi_2(\varphi),$$

where

$$\psi_1(\varphi) = \frac{1}{\pi} \int_0^{2\pi} F(\phi) \left(\sum_{m=0}^{\infty} (-1)^m \sin m\phi \sin m\varphi \right) d\phi \tag{30}$$

and

$$\psi_2(\varphi) = \frac{1}{\pi} \int_0^{2\pi} F(\phi) \left(\sum_{m=0}^{\infty} i(-1)^m \sin\frac{(2m+1)\phi}{2} \sin\frac{(2m+1)\varphi}{2} \right) d\phi. \tag{31}$$

The scattered field Z_s in equation (26) therefore becomes

$$\mathsf{Z}_s \sim \sqrt{\frac{2}{\pi kr}} \, e^{-i(kr-\frac{\pi}{4})} \psi_1(\varphi) + \sqrt{\frac{2}{\pi kr}} \, e^{-i(kr-\frac{\pi}{4})} \psi_2(\varphi). \tag{32}$$

Using the identities

$$(-1)^m \sin m\phi \sin m\varphi = \tfrac{1}{2}\left[\cos m(\phi - \varphi + \pi) - \cos m(\phi + \varphi - \pi)\right]$$

and

$$\sum_{m=0}^{\infty} \cos mx \;=\; \frac{1}{2} \;+\; \sum_{m=-\infty}^{\infty} \pi\delta(x - 2\pi m), \tag{33}$$

the function $\psi_1(\varphi)$ in equation (30) becomes

$$\psi_1(\varphi) = \tfrac{1}{2}\int_0^{2\pi} F(\phi) \sum_{m=-\infty}^{\infty} \Big[\delta(\phi - \varphi + \pi - 2\pi m) - \delta(\phi + \varphi - \pi - 2\pi m)\Big] d\phi$$

$$= \tfrac{1}{2} F(\varphi - \pi) - \tfrac{1}{2} F(-\varphi + \pi). \tag{34}$$

The first term of the scattered field Z_s in equation (32) therefore has a simple relation to the incident field Z_i in equation (24). Poincaré calls the scattered field term with angular dependence $\frac{1}{2}F(\varphi - \pi)$ the *direct* beam, and calls the scattered field term with angular dependence $-\frac{1}{2}F(-\varphi+\pi)$ the *reflected* beam. The incident, direct, and reflected beams are shown in Fig. 36 for the case when the incident beam has uniform amplitude in the angular sector $\alpha < \varphi < \beta$. Poincaré also shows that scattered field would consist only of a direct beam with angular dependence $F(\varphi - \pi)$ if the screen were not present.

Poincaré calls the second term on the right-hand side of equation (32) the *diffracted* beam. To evaluate the angular dependence $\psi_2(\varphi)$ of the diffracted beam, the identity

$$2\sin\frac{(2m+1)\phi}{2}\sin\frac{(2m+1)\varphi}{2} = \cos\frac{(2m+1)(\varphi - \phi)}{2} - \cos\frac{(2m+1)(\varphi + \phi)}{2}$$

and a subsequent integration by parts are used to write equation (31) in the form

$$\psi_2(\varphi) \;=\; \frac{i}{\pi}\int_0^{2\pi} F'(\phi) \sum_{m=0}^{\infty} (-1)^m \frac{1}{2m+1} \sin\frac{(2m+1)(\varphi - \phi)}{2}\, d\phi \;-$$

$$-\; \frac{i}{\pi}\int_0^{2\pi} F'(\phi) \sum_{m=0}^{\infty} (-1)^m \frac{1}{2m+1} \sin\frac{(2m+1)(-\varphi - \phi)}{2}\, d\phi,$$

where the fact that $F(\phi)$ vanishes at $\phi = 0$ and $\phi = 2\pi$ has been used. The closed-form sum

$$\sum_{m=0}^{\infty} (-1)^m \frac{\sin(2m+1)x}{(2m+1)} = \frac{1}{2}\ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right|$$

is now used to obtain

$$\psi_2(\varphi) = \frac{i}{2\pi}\int_0^{2\pi} F'(\phi)\,\ln\left|\frac{\tan\left(\frac{\varphi-\phi+\pi}{4}\right)}{\tan\left(\frac{\varphi+\phi-\pi}{4}\right)}\right| d\phi. \tag{35}$$

For the special case in which the function $F(\varphi)$ that determines the angular dependence of the incident beam is equal to one for $\alpha < \varphi < \beta$ and vanishes outside this interval,

$$F'(\varphi) = \delta(\varphi-\alpha) - \delta(\varphi-\beta), \tag{36}$$

and equation (35) becomes

$$\psi_2(\varphi) = \frac{i}{2\pi}\ln\left|\frac{\tan\left(\frac{\varphi-\alpha+\pi}{4}\right)}{\tan\left(\frac{\varphi-\beta+\pi}{4}\right)}\cdot\frac{\tan\left(\frac{\varphi+\beta-\pi}{4}\right)}{\tan\left(\frac{\varphi+\alpha-\pi}{4}\right)}\right|. \tag{37}$$

Finally, Poincaré considers the case in which the uniform incident beam is narrow, so that $\beta - \alpha$ is small. If equation (37) is written as

$$\psi_2(\varphi) = \frac{i}{2\pi}\ln\left|\frac{\tan\left(\frac{\varphi-\alpha+\pi}{4}\right)}{\tan\left(\frac{\varphi-\alpha+\pi-(\beta-\alpha)}{4}\right)}\cdot\frac{\tan\left(\frac{\varphi+\alpha-\pi+(\beta-\alpha)}{4}\right)}{\tan\left(\frac{\varphi+\alpha-\pi}{4}\right)}\right|,$$

expanding in powers of $\beta - \alpha$ and retaining only the first term gives

$$\psi_2(\varphi) = \frac{i(\beta-\alpha)}{4\pi}\left\{\frac{1}{\cos\frac{\varphi-\alpha}{2}} - \frac{1}{\cos\frac{\varphi+\alpha}{2}}\right\}. \tag{38}$$

Equation (38) is the result of Poincaré to which Sommerfeld refers. It is indeed remarkable that far-field angular dependence $\psi_2(\varphi)$ of the diffracted field in Poincaré's analysis, which is based on the assumption of a narrowly focused incident beam, is identical to the far-field angular dependence of the diffracted field outside the transition strips in Sommerfeld's analysis, which is based on the assumption of an incident plane wave.

A polar plot of $|\psi_2(\varphi)|$, which is proportional to the square root of the optical intensity of Poincaré's diffracted beam, is shown in Fig. 36. As both Poincaré and Sommerfeld note, $\psi_2(\varphi)$ becomes infinite at $\varphi = \pi + \alpha$ and $\varphi = \pi - \alpha$. Sommerfeld makes the comment that the singularities in $\psi_2(\varphi)$ occur inside Sommerfeld's transition parabolas, where the Poincaré result ceases to be valid. Although this is correct, it is also true that the singularities in $\psi_2(\varphi)$ are due only to the specific choice (36) for the derivative of the function $F(\varphi)$ that determines angular dependence of the incident beam. As Poincaré clearly states on page 320 of his paper, the function $\psi_2(\varphi)$ in equation (35) never becomes infinite if $F'(\varphi)$ is finite, as it would be for any realistic incident beam.

In fact, Poincaré's method can be used, with the aid of hindsight, to derive the exact series solutions of Sommerfeld's parallel light (incident plane wave) diffraction problem. A plane wave $e^{ikr\cos(\varphi-\varphi')}$ can be written as the Bessel function expansion

$$e^{ikr\cos(\varphi-\varphi')} = \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(kr)\cos n(\varphi-\varphi'), \tag{39}$$

where $\varepsilon_0 = 1$ and $\varepsilon_n = 2$ for $n > 0$. An elementary derivation of equation (39) is given by Sommerfeld's student Philip McCord Morse (1903–1985) in *Vibration and Sound* (Acoustical Society of America, 1995, p. 192); this is also the simplest case $n = 1$ of Sommerfeld's function u on page 346. If we use the far-field ($kr \to \infty$) approximation for the Bessel function $J_n(kr)$, equation (39) gives

$$\begin{aligned}
e^{ikr\cos(\varphi-\varphi')} &\sim \sum_{n=0}^{\infty} \varepsilon_n i^n \sqrt{\frac{2}{\pi kr}}\,\cos(kr - \frac{\pi}{4} - \frac{n\pi}{2})\cos n(\varphi-\varphi') \\
&\sim \sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\frac{\pi}{4})} \sum_{n=0}^{\infty} \tfrac{1}{2}\varepsilon_n i^n e^{-\frac{in\pi}{2}} \cos n(\varphi-\varphi') + \\
&\quad + \sqrt{\frac{2}{\pi kr}}\,e^{-i(kr-\frac{\pi}{4})} \sum_{n=0}^{\infty} \tfrac{1}{2}\varepsilon_n i^n e^{\frac{in\pi}{2}} \cos n(\varphi-\varphi') \\
&\sim \sqrt{\frac{2}{\pi kr}}\,e^{i(kr-\frac{\pi}{4})} \sum_{n=0}^{\infty} \tfrac{1}{2}\varepsilon_n \cos n(\varphi-\varphi') + \\
&\quad + \sqrt{\frac{2}{\pi kr}}\,e^{-i(kr-\frac{\pi}{4})} \sum_{n=0}^{\infty} \tfrac{1}{2}\varepsilon_n \cos n(\varphi-(\varphi'+\pi)).
\end{aligned}$$

Using equation (33) to evaluate the sums over n, the far-field expression for the plane wave becomes

$$e^{ikr\cos(\varphi-\varphi')} \sim \sqrt{\frac{2}{\pi kr}}\, e^{i(kr-\frac{\pi}{4})}\pi\delta(\varphi-\varphi') + \\ + \sqrt{\frac{2}{\pi kr}}\, e^{-i(kr-\frac{\pi}{4})}\pi\delta(\varphi-(\varphi'+\pi)), \quad 0<\varphi<2\pi. \quad (40)$$

The asymptotic representation (40) is discussed by Morse and Feshbach in *Methods of Theoretical Physics* (McGraw-Hill, 1953, Part II, p. 1379).

The first term on the right-hand side of (40) represents parallel light propagating toward the origin from the direction $\varphi = \varphi'$ and has the form of equation (24), with $F(\varphi) = \pi\delta(\varphi - \varphi')$. If we treat this term as the incident field Z_i in Poincaré's analysis, then the coefficients b_n in equation (28) are

$$b_n = e^{\frac{in\pi}{4}} \sin\frac{n\varphi'}{2}.$$

Using equation (22), the coefficients a_n of the total spatial field Z in equation (17) are $a_n = 2b_n$, so that the total field Z is

$$\mathsf{Z} = \sum_{n=0}^{\infty} 2e^{\frac{in\pi}{4}} \sin\frac{n\varphi'}{2} J_{\frac{n}{2}}(kr) \sin\frac{n\varphi}{2}. \quad (41)$$

Equation (41) is the exact series solution of Sommerfeld's diffraction problem, as can be verified by comparing with Sommerfeld's solution $\mathsf{Z} = u(\varphi') - u(-\varphi')$, where the function u is the series on page 346 for the case $n = 2$.

Poincaré wrote a second paper ("Sur la polarisation par diffraction (seconde partie)," *Acta Mathematica*, Vol. 20, 1897, pp. 313–355) in which he generalizes his analysis to the case in which the incident cylindrical beam is not focused precisely on the edge of the screen. He shows that the essential features of his previous results do not change, and regards Sommerfeld's incident plane wave as a limiting case in which the focal point of the incident beam is infinitely far from the edge of the screen.

49. (page 372) Gustav Robert Kirchhoff (1824–1887) made important contributions to the understanding of electricity and magnetism, mechanics, optics, spectroscopy, and black body radiation. He was the teacher of Ludwig Boltzmann (1844–1906), Heinrich Hertz (1857–1894), and Max Planck (1858–1947). His four-volume *Vorlesungen über mathematische Physik* (*Mechanik*, *Mathematische Optik*, *Electricität und Magnetismus*, *Theorie der*

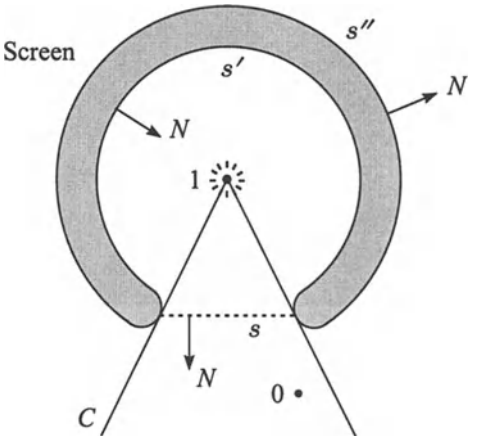


FIG. 37. Geometric configuration for Kirchhoff's diffraction analysis, with a luminous point 1 inside an opaque screen with aperture s (after G. Kirchhoff, *Mathematische Optik*, B. G. Teubner, Leipzig, 1891, p. 80).

Wärme) influenced an entire generation of German physicists. In the *Optik*, Kirchhoff developed a theory of diffraction based directly on a set of partial differential equations for the propagation of light, and thus made a large step toward a unified mathematical analysis of diffraction phenomena. Although Kirchhoff's analysis is approximate and does not use Maxwell's electromagnetic field equations, the fundamental ideas of Kirchhoff's analysis are still used today, and the results of Kirchhoff's theory compare very well with experiments in many optical applications.

We present here the basic principles of Kirchhoff's theory of diffraction, as presented in Lecture 5 of his *Mathematische Optik* (B. G. Teubner, Leipzig, 1891). We then give the specific example in Kirchhoff's Lecture 7 that Sommerfeld uses as a comparison for his exact diffraction solution. Sommerfeld solves only one of the many diffraction problems considered by Kirchhoff, but a comparison of Sommerfeld's analysis with Kirchhoff's shows the striking originality of Sommerfeld's work.

Kirchhoff's diffraction analysis is based on the configuration shown in Fig. 37. A luminous point 1 emits monochromatic light inside an opaque screen with aperture s. Kirchhoff imagines a cone C that touches the screen and has its vertex at point 1. The contact line between the cone and the screen defines the edge of the screen, and this edge divides the surface of the screen into an interior surface s' and an exterior surface s''. It is desired to find the

optical field at the observation point 0 outside the aperture, and in particular at points 0 that lie near the shadow boundary defined by the surface of the cone C.

Kirchhoff assumes that light consists of transverse wave motion in an elastic ether. The field equations that govern the motion of the ether are exactly the equations that are used in current treatments of linear elasticity. The cartesian components of the displacement vector of the ether are denoted by u, v, and w. The displacement component $u(\mathbf{r}, t)$ of a transverse wave in the ether satisfies the wave equation

$$\frac{\partial^2 u}{\partial t^2} = a^2 \Delta u, \tag{42}$$

where a is the speed of light. The same equation holds for the displacement components $v(\mathbf{r}, t)$ and $w(\mathbf{r}, t)$.

The optical field generated by the luminous point 1 in the absence of the screen is assumed to be of the form

$$\begin{aligned} u^*(\mathbf{r}, t) &= \frac{A}{r_1} \cos 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right) + \frac{A'}{r_1} \sin 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right) \\ v^*(\mathbf{r}, t) &= \frac{B}{r_1} \cos 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right) + \frac{B'}{r_1} \sin 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right) \\ w^*(\mathbf{r}, t) &= \frac{C}{r_1} \cos 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right) + \frac{C'}{r_1} \sin 2\pi\left(\frac{r_1}{\lambda} - \frac{t}{T}\right), \end{aligned} \tag{43}$$

where r_1 is the distance from the point 1 to the point of space $\mathbf{r}$, λ is the wavelength of the light, and $T = \lambda/a$ is the period of the light motion. The quantities A, A', B, B', C, and C' are treated here as constants; a more detailed discussion of these quantities and the field generated by the point source 1 is given in Lecture 1, §6 and §7.

To compute the optical field at the observation point 0 in the presence of the screen, Kirchhoff first writes the integral equation

$$u_0 = \frac{1}{4\pi} \int_{s+s''} ds\, \Omega, \tag{44}$$

where

$$\Omega = u\left(\mathbf{r}, t - \frac{r_0}{a}\right) \frac{\partial}{\partial N} \frac{1}{r_0} - \frac{1}{a} \frac{1}{r_0} \frac{\partial r_0}{\partial N} \frac{\partial u}{\partial t}\left(\mathbf{r}, t - \frac{r_0}{a}\right) - \frac{1}{r_0} \frac{\partial u}{\partial N}\left(\mathbf{r}, t - \frac{r_0}{a}\right). \tag{45}$$

In equations (44) and (45), u_0 represents the displacement component u at point 0, r_0 is the distance from the point 0 to a point in the domain of integration, and N is the unit normal shown in Fig. 37. According to Baker and Copson (*loc. cit.*, p. 37), this integral equation for a function u that satisfies equation (42) was first derived by Kirchhoff in 1882; a complete and remarkably modern derivation is given in Lecture 2, §1 of Kirchhoff's *Optik*.

In order to use equation (44), assumptions must be made about the value of the integrand Ω on exterior surface s'' of the screen and on the aperture s. Kirchhoff considers the screen to be a "black body," and makes the following assumptions:

$$\text{(i)} \quad u = \frac{\partial u}{\partial N} = 0 \text{ on } s''$$

$$\text{(ii)} \quad u = u^* \text{ and } \frac{\partial u}{\partial N} = \frac{\partial u^*}{\partial N} \text{ on } s.$$

Kirchhoff's boundary conditions (i) and (ii) have been the subject of much discussion and much criticism (see, for example, Born and Wolf, *loc. cit.*, pp. 378–382 and J. D. Jackson, *Classical Electrodynamics*, Wiley and Sons, 1975, pp. 427–432). As early as 1892, Poincaré showed that Kirchhoff's conditions for u are inconsistent with his conditions for $\frac{\partial u}{\partial N}$ (Baker and Copson, *loc. cit.*, pp. 71–72 and pp. 98–101). Subsequent developments in diffraction theory have removed this inconsistency, but analytic methods for the solution of practical diffraction problems have never strayed far from Kirchhoff's original ideas.

Using Kirchhoff's boundary conditions and the expression for u^* given in equation (43), equations (44) and (45) for the displacement component u_0 become

$$u_0 = \frac{1}{4\pi} \int_s ds\, \Omega, \tag{46}$$

where

$$\begin{aligned}\Omega = {} & \frac{A}{r_1 r_0}\left(\frac{1}{r_1}\frac{\partial r_1}{\partial N} - \frac{1}{r_0}\frac{\partial r_0}{\partial N}\right)\cos 2\pi\left(\frac{r_1 + r_0}{\lambda} - \frac{t}{T}\right) + \\ & + \frac{2\pi A}{r_1 r_0 \lambda}\left(\frac{\partial r_1}{\partial N} - \frac{\partial r_0}{\partial N}\right)\sin 2\pi\left(\frac{r_1 + r_0}{\lambda} - \frac{t}{T}\right) + \\ & + \frac{A'}{r_1 r_0}\left(\frac{1}{r_1}\frac{\partial r_1}{\partial N} - \frac{1}{r_0}\frac{\partial r_0}{\partial N}\right)\sin 2\pi\left(\frac{r_1 + r_0}{\lambda} - \frac{t}{T}\right) - \\ & - \frac{2\pi A'}{r_1 r_0 \lambda}\left(\frac{\partial r_1}{\partial N} - \frac{\partial r_0}{\partial N}\right)\cos 2\pi\left(\frac{r_1 + r_0}{\lambda} - \frac{t}{T}\right). \end{aligned} \tag{47}$$

Kirchhoff now makes three further assumptions. First, he assumes that the distances r_0 and r_1 in equation (47) are large compared to the wavelength λ. Second, he assumes that the aperture s is small, so that r_1, r_0, $\frac{\partial r_1}{\partial N}$ and $\frac{\partial r_0}{\partial N}$ can be treated as constants where they appear in Ω outside of the arguments of the trigonometric functions. Third, he considers only points 0 such that the equality

$$\frac{\partial r_0}{\partial N} = -\frac{\partial r_1}{\partial N}$$

is nearly satisfied. This implies that the difference $\left(\frac{\partial r_1}{\partial N} - \frac{\partial r_0}{\partial N}\right)$ in Ω cannot vanish, and that the terms proportional to $1/(r_1^2 r_0)$ and $1/(r_1 r_0^2)$ in Ω can be safely neglected compared to the terms proportional to $1/(r_1 r_0 \lambda)$. (Geometrically, the condition $\frac{\partial r_0}{\partial N} \approx -\frac{\partial r_1}{\partial N}$ means that the line segments from point 1 to the small aperture are nearly parallel to the line segments from point 0 to the small aperture, or, in other words, that the *angle of diffraction* is small.) With these assumptions, equations (46) and (47) give

$$u_0 = \frac{1}{\lambda r_1 r_0}\frac{\partial r_1}{\partial N}\left(A\int_s ds\ \sin 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right) - \right.$$
$$\left. - A'\int_s ds\ \cos 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right)\right). \qquad (48)$$

Analogous considerations for the displacement components v and w give the expressions

$$v_0 = \frac{1}{\lambda r_1 r_0}\frac{\partial r_1}{\partial N}\left(B\int_s ds\ \sin 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right) - \right.$$
$$\left. - B'\int_s ds\ \cos 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right)\right), \qquad (49)$$

$$w_0 = \frac{1}{\lambda r_1 r_0}\frac{\partial r_1}{\partial N}\left(C\int_s ds\ \sin 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right) - \right.$$
$$\left. - C'\int_s ds\ \cos 2\pi\left(\frac{r_1+r_0}{\lambda} - \frac{t}{T}\right)\right). \qquad (50)$$

Kirchhoff next calculates the time-averaged optical intensity J_0 at the point 0. This intensity is defined as

$$J_0 = \frac{1}{T}\int_0^T dt\ (u_0^2 + v_0^2 + w_0^2). \tag{51}$$

The optical intensity is the quantity that is assumed to be sensed by the eye or recorded on a photographic plate. In the elastic theory, the intensity is proportional to the kinetic energy density of a time-harmonic field. Substitution of equations (48), (49), and (50) into equation (51) gives, after a pleasant calculation,

$$J_0 = J^* \frac{1}{\lambda^2 r_0^2}\left(\frac{\partial r_1}{\partial N}\right)^2 (c^2 + s^2), \tag{52}$$

where

$$c = \int_s ds\ \cos 2\pi\left(\frac{r_1 + r_0}{\lambda}\right)$$

$$s = \int_s ds\ \sin 2\pi\left(\frac{r_1 + r_0}{\lambda}\right)$$

and J^* is the optical intensity

$$\begin{aligned} J^* &= \frac{1}{T}\int_0^T dt\ (u^{*2} + v^{*2} + w^{*2}) \\ &= \frac{1}{2r_1^2}(A^2 + A'^2 + B^2 + B'^2 + C^2 + C'^2) \end{aligned}$$

of the luminous point 1 at the aperture of the screen. The quantites c and s can also be computed from the expressions

$$c = \int_s ds\ \cos\left(2\pi\frac{r_1 + r_0}{\lambda} + \delta\right) \tag{53}$$

$$s = \int_s ds\ \sin\left(2\pi\frac{r_1 + r_0}{\lambda} + \delta\right), \tag{54}$$

since the introduction of the arbitrary constant δ does not change the value of $c^2 + s^2$.

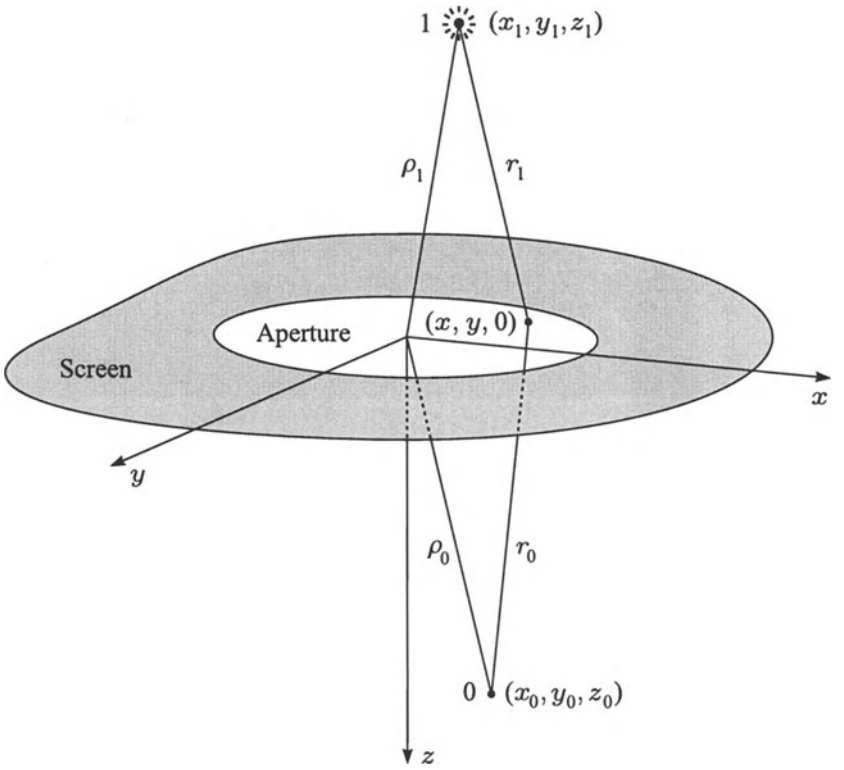


FIG. 38. Coordinate system for Kirchhoff's diffraction analysis, with distances r_0, r_1, ρ_0, and ρ_1.

Kirchhoff gives several examples of the application of equation (52) to specific diffraction problems. In the examples, a Cartesian coordinate system is chosen so that the aperture is in the x,y plane, with the origin of the coordinate system at some point in the aperture. The luminous point 1 has coordinates (x_1, y_1, z_1) and the observation point 0 has coordinates (x_0, y_0, z_0), as shown in Fig. 38. With this choice of coordinates, the distances r_1 and r_0 in equations (53) and (54) are

$$r_1 = \sqrt{(x_1 - x)^2 + (y_1 - y)^2 + z_1^2}$$

and

$$r_0 = \sqrt{(x_0 - x)^2 + (y_0 - y)^2 + z_0^2}.$$

In order to evaluate the integrals c and s in equations (53) and (54), the quantity $r_1 + r_0$ is expanded in powers of the integration variables x and y in

the plane of the aperture. If the distances ρ_1 and ρ_0 are defined by

$$\rho_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}, \qquad \rho_0 = \sqrt{x_0^2 + y_0^2 + z_0^2},$$

then the expansion of $r_1 + r_0$ to second order in x and y is

$$r_1 + r_0 = \rho_1 + \rho_0 - x\left(\frac{x_1}{\rho_1} + \frac{x_0}{\rho_0}\right) - y\left(\frac{y_1}{\rho_1} + \frac{y_0}{\rho_0}\right) + \\ + \frac{x^2 + y^2}{2}\left(\frac{1}{\rho_1} + \frac{1}{\rho_0}\right) - \frac{(xx_1 + yy_1)^2}{2\rho_1^3} - \frac{(xx_0 + yy_0)^2}{2\rho_0^3}. \quad (55)$$

Kirchhoff uses the term *Fresnel diffraction* to describe diffraction phenomena whose analysis depends on retaining both the first and second-order terms in x and y in the expansion of $r_1 + r_0$. Diffraction phenomena that can be analyzed using only the first-order terms in x and y are called *Fraunhofer* diffraction.*

It is plausible that a truncated expansion of $r_1 + r_0$ in powers of the integration variables x and y in the plane of the aperture gives a good approximation to the integrals c and s when the dimensions of the aperture are small compared to the distances ρ_1 and ρ_0. However, Kirchhoff states explicitly and with emphasis that he will use such an expansion even when the dimensions of the aperture are large compared to the distances ρ_1 and ρ_0. He claims, with reference to an earlier analysis in Lecture 4, that the portions of the aperture for which the truncated expansion is not accurate give negligible contributions to the integrals c and s in equations (53) and (54).

Kirchhoff considers the problem of diffraction by a half plane in detail, and this is the example to which Sommerfeld refers in his paper. The geometric configuration of the problem is illustrated in Fig. 39. The aperture is now the entire half plane $z = 0$, $x > 0$. The luminous point 1 is on the negative z-axis, directly above the edge of the diffracting screen. For the configuration in Fig. 39, the coordinates x_1 and y_1 of the luminous point vanish. Also, Kirchhoff considers only points 0 for which the squares and products of x_0/ρ_0 and y_0/ρ_0 in equation (55) may be neglected. The optical intensity J_0 at point 0 is then

$$J_0 = K(c^2 + s^2),$$

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*Joseph Fraunhofer (1787–1826) invented the diffraction grating and was the founder of modern spectroscopy. He began as an apprentice to a glass cutter, and became the director of the Untzschneider Optical Institute in Munich. His work there set the stage for the long preeminence of the German optical industry.

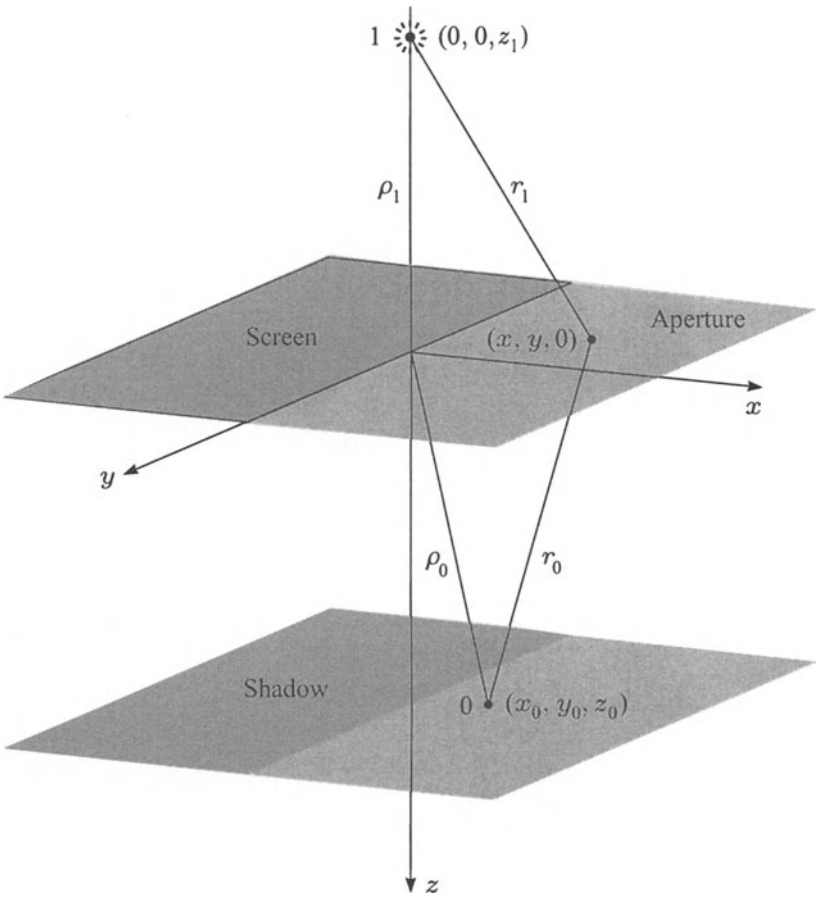

FIG. 39. Geometric configuration for the diffraction of light from the luminous point 1 by a semi-infinite planar screen.

where K is a constant and

$$c = \int_0^\infty dx \int_{-\infty}^\infty dy \cos\left\{\frac{2\pi}{\lambda}\left[\rho_1 + \rho_0 + \frac{x^2+y^2}{2}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right) - \right.\right.$$
$$\left.\left. - x\left(\frac{x_0}{\rho_0}\right) - y\left(\frac{y_0}{\rho_0}\right)\right] + \delta\right\}$$

$$s = \int_0^\infty dx \int_{-\infty}^\infty dy \sin\left\{\frac{2\pi}{\lambda}\left[\rho_1 + \rho_0 + \frac{x^2+y^2}{2}\left(\frac{1}{\rho_0}+\frac{1}{\rho_1}\right) - \right.\right.$$
$$\left.\left. - x\left(\frac{x_0}{\rho_0}\right) - y\left(\frac{y_0}{\rho_0}\right)\right] + \delta\right\}.$$

With the change of variables

$$\xi = \sqrt{\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left(x - \frac{\dfrac{x_0}{\rho_0}}{\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}}\right)$$

$$\eta = \sqrt{\frac{\pi}{\lambda}\left(\frac{1}{\rho_1}+\frac{1}{\rho_0}\right)}\left(y - \frac{\dfrac{y_0}{\rho_0}}{\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}}\right)$$

and the choice

$$\delta = \frac{2\pi}{\lambda}\left(-\rho_1 - \rho_0 + \frac{\left(\dfrac{x_0}{\rho_0}\right)^2 + \left(\dfrac{y_0}{\rho_0}\right)^2}{2\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_0}\right)}\right),$$

the quantities c and s and the optical intensity J_0 become

$$\begin{aligned} c &= \frac{\lambda}{\pi}\frac{\rho_1\rho_0}{\rho_1+\rho_0}\int_{-\infty}^{\infty} d\eta \int_u^{\infty} d\xi\ \cos(\xi^2+\eta^2) \\ &= \frac{\lambda}{\pi}\frac{\rho_1\rho_0}{\rho_1+\rho_0}\sqrt{\frac{\pi}{2}}\int_u^{\infty} d\xi\ (\cos\xi^2 - \sin\xi^2) \end{aligned}$$

$$\begin{aligned} s &= \frac{\lambda}{\pi}\frac{\rho_1\rho_0}{\rho_1+\rho_0}\int_{-\infty}^{\infty} d\eta \int_u^{\infty} d\xi\ \sin(\xi^2+\eta^2) \\ &= \frac{\lambda}{\pi}\frac{\rho_1\rho_0}{\rho_1+\rho_0}\sqrt{\frac{\pi}{2}}\int_u^{\infty} d\xi\ (\cos\xi^2 + \sin\xi^2), \end{aligned}$$

$$J_0 = K\frac{\lambda^2}{\pi}\left(\frac{\rho_1\rho_0}{\rho_1+\rho_0}\right)^2\left[\left(\int_u^{\infty} d\xi\ \cos\xi^2\right)^2 + \left(\int_u^{\infty} d\xi\ \sin\xi^2\right)^2\right] \quad (56)$$

$$= K\frac{\lambda^2}{\pi}\left(\frac{\rho_1\rho_0}{\rho_1+\rho_0}\right)^2\left[M^2(u)+N^2(u)\right], \quad (57)$$

where

$$M(u) = \int_u^\infty d\xi \ \cos(\xi^2 - u^2) \tag{58}$$

$$N(u) = \int_u^\infty d\xi \ \sin(\xi^2 - u^2) \tag{59}$$

and

$$u = -x_0 \sqrt{\frac{\pi \rho_1}{\lambda \rho_0 (\rho_1 + \rho_0)}}.$$

With the approximations that have already been made, $\rho_0 \approx z_0$ and $\rho_1 \approx -z_1$, so that the quantity u can also be written as

$$u = -x_0 \sqrt{\frac{\pi z_1}{\lambda z_0 (z_1 - z_0)}}. \tag{60}$$

Equation (57) is the result of Kirchhoff that Sommerfeld quotes on the bottom of page 372, and equation (60) is Kirchhoff's equation (11a) that Sommerfeld refers to at the top of page 373. To relate Sommerfeld's results to Kirchhoff's, we refer to Fig. 40 and set

$$z_1 = -\infty, \quad x_0 = r \sin(\varphi - \varphi' + \pi), \quad z_0 = r \cos(\varphi - \varphi' + \pi).$$

The polar angle φ in Fig. 40 is measured counterclockwise from the diffracting screen S, as in Sommerfeld's Fig. 5. Setting $z_1 = -\infty$ is equivalent to the assumption that the incident light is a plane wave. Sommerfeld's first equations on page 373 obviously contain misprints.

As Sommerfeld shows by comparison with his exact solution, Kirchhoff's result (57) for the optical intensity J_0 is accurate when the angle of diffraction $\varphi - \varphi' + \pi$ is small. In fact, equation (57) can be used to predict the locations of the fringes that are observed experimentally near the shadow boundary $x = 0$, $z < 0$. The fringes correspond to the maxima and minima of the intensity J_0. The values of u for which J_0 is a maximum or minimum satisfy the equation

$$\frac{dJ_0}{du} = 0,$$

which by equations (56) and (58) is equivalent to

$$\begin{aligned} 0 &= \cos u^2 \int_u^\infty d\xi \ \cos \xi^2 + \sin u^2 \int_u^\infty d\xi \ \sin \xi^2 \\ &= M(u). \end{aligned} \tag{61}$$

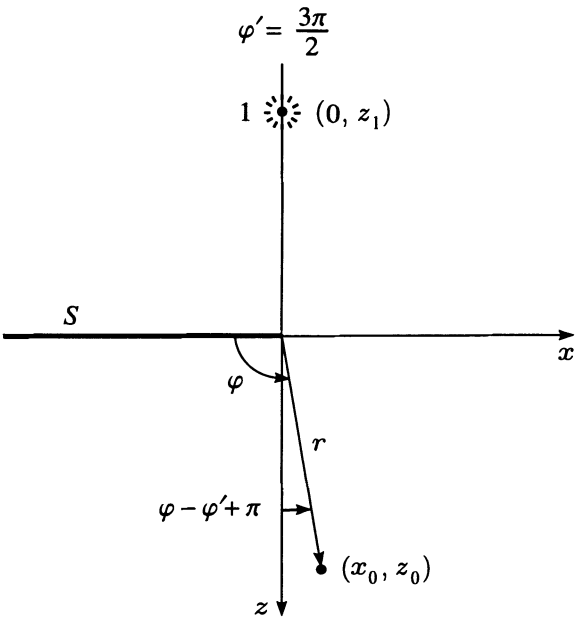

FIG. 40. Angle of diffraction $\varphi - \varphi' + \pi$ for the diffraction of light from the luminous point 1 by the semi-infinite planar screen S.

Kirchhoff shows analytically that $M(u)$ has no zeros for $u > 0$. Thus the optical intensity J_0 has no maxima or minima for $x_0 < 0$, and there are no fringes in the shadow region. To calculate the negative values of u for which $M(u) = 0$, Kirchhoff uses the identity

$$M(u) = \sqrt{\frac{\pi}{2}}\,(\cos u^2 + \sin u^2) - M(-u).$$

Equation (61) thus becomes

$$\sin\left(u^2 + \frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}} M(-u). \tag{62}$$

Since $M(-u)$ vanishes for large negative values of u, the solutions of equation (62) for large negative values of u are of the form

$$u = -\sqrt{\left(n - \frac{1}{4}\right)\pi},$$

where n is an integer. Numerical calculations show that even the solutions of equation (62) closest to zero differ but little from this form. Kirchhoff quotes a calculation of Fresnel (who obtained equation (62) by a different method) that gives the following locations for the first two maxmima and minima of J_0:

$$\begin{aligned} u &= -\sqrt{0.741\pi}, && \text{first maximum} \\ u &= -\sqrt{1.754\pi}, && \text{first minimum} \\ u &= -\sqrt{2.749\pi}, && \text{second maximum} \\ u &= -\sqrt{3.751\pi}, && \text{second minimum.} \end{aligned}$$

We find that the first minimum is closer to $u = -\sqrt{1.753\pi}$; we otherwise agree with Fresnel's results to the given three decimal place accuracy. For $z_1 = -\infty$, $\lambda = 5 \times 10^{-4}$ mm, and $r = 1$ m, the angle of diffraction ϑ corresponding to the second minimum of J_0 is $\vartheta \approx 10^{-3}$. For such small angles of diffraction, the difference between Kirchhoff's results and Sommerfeld's is entirely negligible.

50. (page 374) Hermann Rudolf Eugen Maey (1869–1946) received the Ph.D. degree from the University of Königsberg in 1893. He spent most of his career as a secondary school teacher of physics in Remscheid and Bonn. The paper that Sommerfeld cites ("Über die Beugung des Lichtes an einem geraden, scharfen Schirmrande," *Annalen der Physik und Chemie* (*Wiedemann's Annalen*), Vol. 49, 1893, pp. 69–104) is based on Maey's doctoral dissertation, a theoretical and experimental investigation of the light emitted by the apparent luminous line at a diffracting knife edge.

51. (page 374) This hope of Sommerfeld was never realized. In his *Optics* (Academic Press, New York, 1954, p. 253; Volume IV of the *Lectures on Theoretical Physics* that Sommerfeld compiled at the end of his life), Sommerfeld writes that "it has been possible to construct these [branched] solutions only for the simplest case of the half plane."

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Appendix I

The History and Present State of Discoveries relating to Vision, Light and Colours

Joseph Priestley

London, 1772, pp. 171–182

PERIOD IV. *Discoveries in opticks from the time of Descartes to that of Sir Isaac Newton.*

SECTION IV.

EXPERIMENTS CONCERNING THE INFLECTION OF LIGHT.

In the title of this section there occurs a new term in the doctrine of light, corresponding to a new property of the rays, discovered within this period; and though, in fact, it seems to be nothing more than an effect of the same power which is the cause of reflexion and refraction, particularly circumstanced, yet it is such an effect as could hardly have been suspected without actual observation. Had Descartes, or any other philosopher before this period, been asked, what would have become of a ray of light, which should pass as near as possible to any body, but without actually impinging on it; he would have replied, without hesitation, that it would proceed in a straight line, without being affected by its approach. But it was now found, that light coming within a certain distance of any body, will either be bent from it, or towards it, which is a kind of imperfect reflexion or refraction. This property was first called *diffraction*, but has now generally obtained the name of *inflection*, which is the term adopted by Sir Isaac Newton, whose experiments on this subject, though not completed, are peculiarly striking and important.

The first who made this discovery concerning the diffraction, or inflection of light, was Father Grimaldi. At least, he first published an account of it, in his treatise *De Lumine, coloribus, et iride*, printed in 1666; for I do not

find that any other person lays claim to the discovery, except Dr. Hooke, who communicated his observations on this subject to the royal society in 1672, some months after the first communication of Sir Isaac Newton's discovery, concerning the different refrangibility of the rays of light; and, as it should seem, in order to fulfil the promise which he thereupon made, to bring some objections to Sir Isaac Newton's doctrine, at the next meeting of the society. It appears, however, that Dr. Hooke had not heard of the discoveries of Grimaldi; for he speaks of his own, as the discovery of a "new property of light not mentioned by any optical writer before him."* Indeed, the circumstances of Dr. Hooke's experiments are so different from those of Grimaldi, that his account is in itself extremely probable.

Considering the observations of Dr. Hooke, as being made independent of those of Grimaldi, and perhaps about the same time, and being far less perfect than his, I shall begin with them, as proper to introduce the more accurate and important observations of the Italian philosopher; which, indeed, do well deserve to be recited at some length, no writer that I have met with having done them justice.

Dr. Hooke, having made his room completely dark, he admitted into it a beam of the sun's light, by a very small hole in a brass plate, fixed in the window shutter. This beam, spreading itself, formed a cone, the apex of which was in the hole, and the base was on a paper, so placed, as to receive it at some distance. In this image of the sun, thus painted on the paper, he observed that the middle was much brighter than the edges, and that there was a kind of a dark penumbra about it, of about a 16th part of the diameter of the circle; which penumbra, he says, must be ascribed to a property of light, which he promised to explain. Having observed this, at about the distance of two inches from the former, he let in another cone of light; and receiving the bases of them, at such a distance from the holes, as that the circles intersected each other, he observed that there was not only a penumbra, or darker ring, encompassing the lighter circle, but a manifest dark line, or circle, which appeared even where the limb of the one interfered with that of the other. This appearance is distinctly represented, fig. 52.

Comparing the diameter of this base, with its distance from the hole, he found it to be by no means the same as it would have been, if it had been formed by straight lines, drawn from the extremities of the sun's disk, but varied with the size of the holes, and the distance of the paper.

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*Birch's History, Vol. 3, p. 10.

Struck with this appearance, he proceeded to make farther experiments concerning the nature of light thus transmitted. To give a just idea of which, he held an opaque body, BB, fig. 53, as to intercept the light that entered at a hole in the window shutter O, and was received on the screen AP. In these circumstances, he observed, that the shadow of the opaque body (which was a round piece of wood, not bright or polished) was all over somewhat enlightened; but more especially towards the edge. Some persons, who were present, imagining that this light within the shadow might be produced by some kind of reflexion from the side of the opaque body, on account of its roundness; and others supposing it might proceed from some reflexion from the sides of the hole, in the piece of brass through which the light was admitted into the room; to obviate both these objections, he admitted the light through a hole burnt in a piece of pasteboard, and intercepted it with a rasor, which had a very sharp edge; but still the appearances were the very same as before; so that, upon the whole, he concluded that they were occasioned by a new property of light, different from any that had been observed by preceding writers.

He farther diversified this experiment, by placing the rasor so as to divide the cone of light into two parts, the hole in the shutter remaining as before, and placing the paper so as that none of the enlightened part of the circle fell upon it, but only the shadow of the rasor; and, to his great surprise, he observed what he calls a *very brisk and visible radiation*, striking down upon the paper, of the same breadth with the diameter of the lucid circle; and this radiation always struck perpendiculary from the line of shadow, and, like the tail of a comet, extended more than 10 times, and probably more than 100 times the breadth of the remaining part of the circle; nay, as far as he could find, by many trials, the light from the edge struck downward into the shadow very near to a quadrant, though the greater were the deflexions of this new light from the direct radiations of the cone, the more faint they were.

Observing this appearance with more attention, he found wherever there was a part of the interposed body higher than the rest, that is, opposite to it, the radiation of light into the shadow was brighter, as in the figure; and wherever there was a notch or gap in it, there would be a dark stroke in the half enlightened shadow. From all these appearances he concluded, that they were to be ascribed to a new property of light, whereby it is deflected from straight lines, contrary to what had been before asserted by optical writers.

It does not appear, however, that our philosopher ever prosecuted this experiment to any purpose; as all that we find of his on the subject of light, after this time, are some crude thoughts, which he read at a meeting of the royal society,

on the 18th of March 1675, which however, as they are only short hints, I shall copy.

They consist of eight articles, and, as he thought, contained an account of several properties of light, that had not been noticed before. 1. There is a *deflection* of light, differing both from reflexion and refraction, and seeming to depend on the unequal density of the constituent parts of the ray, whereby the light is dispersed from the place of condensation, and rarified, or gradually diverged into a quadrant. 2. This deflection is made towards the superficies of the opaque body perpendicularly. 3. Those parts of the diverged radiations which are deflected by the greatest angle from the straight or direct radiations are the faintest, and those that are deflected by the least angles are the strongest. 4. Rays cutting each other in one common foramen do not make the angles at the vertex equal. 5. Colours may be made without refraction. 6. The diameter of the sun cannot be truly taken with common sights. 7. The same rays of light, falling upon the same point of an object, will turn into all sorts of colours, by the various inclination of the object. 8. Colours begin to appear when two pulses of light are blended so well, and so near together, that the sense takes them for one.*

After this view of the ingenious, but imperfect experiments of Dr. Hooke, I proceed to relate the more valuable discoveries of Father Grimaldi. Having introduced a ray of light, through a very small hole, AB, fig. 54, into a darkened room, he observed that the light was diffused into the form of a cone, the base of which was CD; and that if any opaque body, FE, was placed in this cone of light, at a considerable distance from the hole, and the shadow was received upon a piece of white paper, the boundaries of it were not confined within GH, or the penumbra IL, occasioned by the light proceeding from different parts of the aperture, and of the disk of the sun, but extended to MN, at which he was very much surprised, suspecting, and finding by calculation, that it was considerably broader than it could have been made by rays passing in right lines by the edges of the object.

But the most remarkable circumstance in this appearance was that, upon the lucid part of the base, CM and NC, streaks of coloured light were plainly distinguished, each being terminated by blue on the side next to the shadow, and by red on the other; and though these coloured streaks depended, in some measure, on the size of the aperture AB, because they could not be made to

*Hooke's Posthumous Works, p. 190.

appear if it was large, yet he found that they were not limited either by it, or by the diameter of the sun's disk.

He farther observed, that these coloured streakes were not all of the same breadth, but grew narrower as they receded from the shadow, and were each of them broader, the farther the shadow was received from the opaque body, and also the more obliquely the paper on which they were received was held with respect to it. He never observed more than three of these streaks.

To give a clearer idea of these coloured streaks, he drew the representation of them, exhibited in fig. 55, in which NMO represents the broadest and most luminous streak, next to the dark shadow X. In the space in which M is placed there was no distinction of colour, but the space NN was blue, and the space OO, on the other side of it, was red. The second streak, QPR, was narrower than the former; and of the three parts of which it consisted, the space P had no particular colour, but QQ was a faint blue, and RR a faint red. The third streak, TSV, was exactly similar to the two others, but narrower than either of them, and the colours still fainter.*

These coloured streaks he observed to lie parallel to the shadow of the opaque body; but when it was of an angular form, they did not make the same acute angles, but were bent into a curve, the outermost being rounder than those that were next the shadow, as is represented, fig. 56. If it was an inward angle, as DCH, the coloured streaks, parallel to each of the two sides, crossed without obliterating one another; only the colours were thereby rendered more intense, or mixed.

The light that formed these coloured streaks, my reader will perceive, must have been bent *from* the body; but this attentive observer has likewise given an account of other appearances, which must have been produced by the light bending *towards* the body. For within the shadow itself he sometimes perceived coloured streaks, similar to those above mentioned on the outside of the shadow. Sometimes he saw more of them, and sometimes fewer; but for this purpose a very strong light was requisite, and the opaque body was obliged to be long, and of a moderate breadth; which, he says, is easily found by experience. A hair, for instance, or a fine needle, did not answer so well as a thin and narrow plate; and the streaks were most distinguishable when the shadow was taken at the greatest distance; but then the light grew fainter in the same proportion.

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*Grimaldi de Lumine, p. 2.

The number of these streaks within the shadow was greater in proportion to the breadth of the plate. They were, at least, two, and sometimes four, if a thicker rod were made use of. But, with the same plate or rod, more or fewer streaks appeared, in proportion to the distance at which the shadow was received; but they were broader when they were fewer, and narrower when there were more of them; and they were all much more distinct when the paper was held obliquely.

These coloured streaks within the shadow, like those on the outside of it, were bent into an arch, round the acute angles of the shadow, as they are represented, fig. 57. At this angle also, as at D, other shorter lucid streaks were visible, bent in the form of a plume, as they are drawn betwixt D and C, each bending round and meeting again in D. These angular streaks appeared, though the plate or rod was not wholly immersed in the beam of light, but the angle of it only; and there were more or fewer in number, in proportion to the breadth of the rod or plate. If the plate or rod was very thin, the coloured streaks within the shadow might be seen to bend round from the opposite sides, and meet one another, as at B. A only represents a section of the figure, and not a proper termination of the shadow, and the streaks within each side of it. The coloured streaks without the shadow, he also observes, bend round it, in the same manner.

Our author acknowledges, that he omits several observations of less consequence, which cannot but occur to any person who shall make the experiment; and he says, that he was not able to give a perfectly clear idea of what he has attempted to describe, nor does he think it in the power of words to do it.

In order to obtain the more satisfactory proof that rays of light do not always proceed in straight lines, but really bend, in passing by the edges of bodies, he diversified the first of the above mentioned experiments in the following manner. He admitted a beam of light, by a very small aperture, into a darkened room, as before; and at a great distance from it, he fixed a plate EF, fig. 58, with a small aperture, GH, which admitted only a part of the beam of light, and found, that when the light transmitted through this plate, was received at some distance, upon a white paper, the base IK was considerably larger than it could possibly have been made by rays issuing in right lines through the two apertures, as the other straight lines drawn close to their edges plainly demonstrate.*

*Grimaldi de Lumine, p. 9.

That those who chuse to repeat these experiments may not be disappointed in their expectations from them, our author gives the following more particular instructions. The sun's light must be very intense, and the aperture through which it is transmitted very narrow, particularly the first, CD, and the white paper, IK, on which it is received, must be at a considerable distance from the hole GH; otherwise it will not exceed NO, which would be the breadth of the beam of light proceeding in straight lines. He generally made the aperture CD $\frac{4}{300}$ or $\frac{5}{300}$ parts of an ancient Roman foot, and the second aperture, GH, $\frac{25}{300}$ or $\frac{30}{300}$; and the distances DG and GN were, at least, 12 such feet. The observation was made in the summer time, when the atmosphere was free from all vapours, and about mid-day.*

F. Grimaldi also made the same experiment that has been recited from Dr. Hooke, in which two beams of light, entering a darkened room by two small apertures near to one another, projected cones of light; which, at a certain distance, in part coincided; and he particularly observed that the dark boundaries of each of them were visible within the lucid ground of the other.†

To these discoveries of Grimaldi, I shall subjoin an additional observation of Dechales; who took notice, that if small scratches be made in any piece of polished metal, and it be exposed to the beams of the sun, in a darkened room, it will reflect the rays streaked with colours, in the direction of the scratches; as will appear if the reflected light be received upon a piece of white paper.§ That these colours are not produced by refraction, he says, is manifest; for that, if the scratches be made upon glass, the effect will be the same; and in this case, if the light had been refracted at the surface of the glass, it would have been transmitted through it.‡ From these, and many other observations, he concludes that colour does not depend upon the refraction of light only, nor upon a variety of other circumstances, which he particularly enumerates, and the effects of which he discusses, but upon the intensity of the light only.‖

The very imperfect views and conclusions of the philosophers of this period exhibit an amusing and instructive prospect; as they demonstrate that it is by no means necessary to have just views, and a true hypothesis, *a priori*, in order to make real discoveries. Very lame and imperfect theories are sufficient to

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*Grimaldi de Lumine, p. 10.

†Ib. p. 187.

§Cursus, Vol. 3, p. 726.

‡Ib. p. 738.

‖Ib. p. 739.

suggest useful experiments, which serve to correct those theories, and give birth to others more perfect. These, then, occasion farther experiments, which bring us still nearer the truth; and in this method of *approximation*, we must be content to proceed, and we ought to think ourselves happy, if, in this slow method, we make any real progress. As an historian, I am particularly pleased, that the philosophers of the period of which I am now treating, published their works before their theories were exploded. Had the case been otherwise, we should probably never have heard of them; whereas, at present, we are better able to enter into their views, and to observe the connection between their hypotheses and their experiments.

The professed object of Father Grimaldi's whole book, is to determine the great question of those times, viz. whether light be a substance, or a quality; and after discussing it very largely, in a close printed quarto, consisting of 535 pages, he concludes, with the Aristotelians, that light is no real substance, but only a mode or property of bodies; or, rather in his own words, it is not a *substantial*, but an *accidental quality*. But it is not my business to note the mistakes of great men, but to record their useful labours.

I shall close this section with an account of a phenomenon of vision observed by M. De la Hire, because the subject of this section, viz. the *inflection of light*, seems to supply the true solution of it, though the author himself thought otherwise. It is observable, he says, that when we look at a candle, or any luminous body, with our eyes nearly shut, rays of light are extended from it, in several directions, to a considerable distance, like the tails of comets. This appearance exercised the sagacity of Descartes, and Rohault, as well as of our author; but all three seem to have been mistaken with respect to it. Descartes ascribed this effect to certain wrinkles in the surface of the humours of the eye. Rohault says that, when the eye-lids are nearly closed, the edges of them act like convex lenses; but our author says, that the moisture on the surface of the eye, adhering partly to the eye itself and partly to the edge of the eye lid, makes a concave mirror, and so disperses the rays at their entrance into the eye. But the true reason seems to be, that the light, passing among the eyelashes, in this situation of the eye, is inflected by its near approach to them, and therefore enters the eye in a great variety of directions. The two former of these opinions are particularly stated and objected to by our author.†

† Grimaldi de Lumine, p. 392.

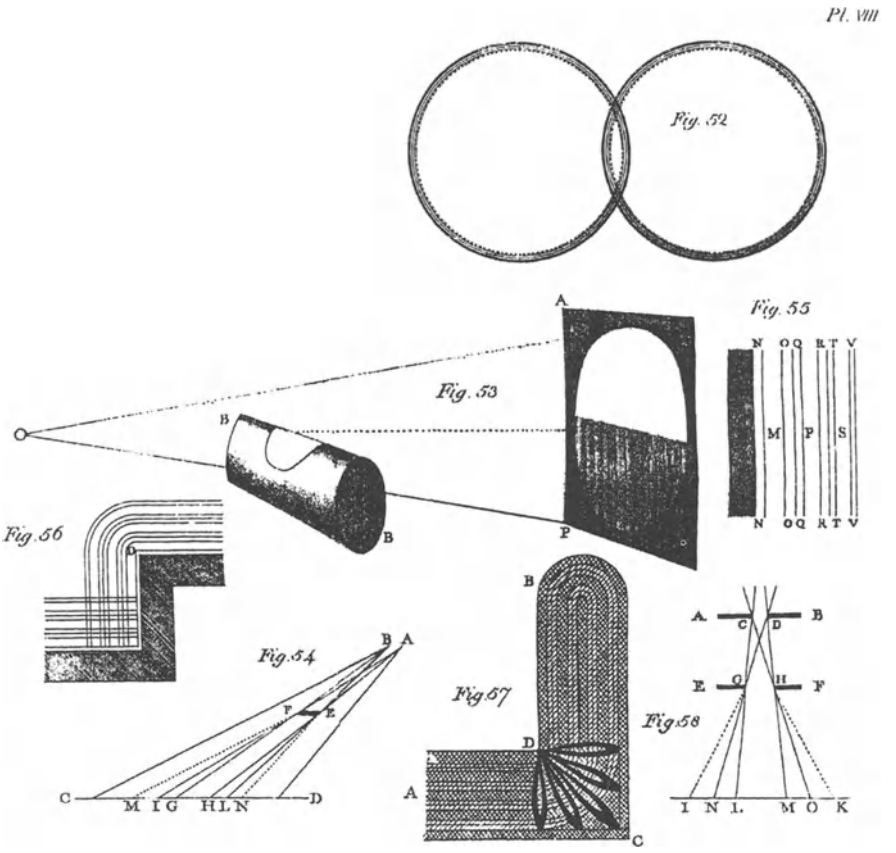

FIG. I-1. Plate VIII of *The History and Present State of Discoveries relating to Vision, Light and Colours*, Joseph Priestley, London, 1772.

Appendix II

On the Mathematical Theory of Diffraction Phenomena

Arnold Sommerfeld

Nachrichten von der Königlichen Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, Nr. 4, 1894, pp. 338–342

(presented by F. Klein at the meeting of December 8, 1894)

Physically expressed, the theme of the following communication is *diffraction theory*; mathematically expressed, it would be called *the integration of the partial differential equation* $\Delta u + k^2 u = 0$ *on a Riemann surface*. Since I postpone all details to a later publication, I wish here only to explain in essence to what extent the two problems are the same.

We consider the diffraction phenomena which are caused by an arbitrary screen S of vanishing thickness. We mean not only *the diffraction of light*, but also *the diffraction of electrical waves*, and in fact the latter in particular, since the usual theory for these waves fails completely for larger wavelengths, while our theory directly yields especially appropriate expressions. Moreover, our theory also extends to *acoustic waves*; but since the presentation of the theory for these waves must be somewhat modified, we exclude them for the sake of brevity. We will borrow the terminology of optics.

We consider monochromatic light of oscillation period τ and a stationary state. Instead of the actual components of light, we can consider three functions U, V, W which are independent of t and from which the components in the x, y, z directions are calculated as $\Re e\,(e^{2\pi it/\tau}U)$, $\Re e\,(e^{2\pi it/\tau}V)$, $\Re e\,(e^{2\pi it/\tau}W)$. We assume that the screen is a perfect conductor for the considered wave motion. Then the components that fall in surface of the screen must vanish. This condition holds as S is approached from both sides. The components themselves are thus continuous in the passage across the screen. But for the normal derivatives this is not initially prescribed. These derivatives will in general undergo a jump. *Conceived as single-valued functions of space, the functions* U, V, W *lose their analytic character at points of* S.

Whenever nonanalytic functions of this type are treated, one forms their analytic continuation, and thus attains a better understanding of the nature of

the functions. *We assert that the continuations formed from one side and the other will be identical if the screen is planar.*

Let the plane of the screen be the Y, Z plane; we denote its boundary curve by C. In this case the following conditions for W obtain: a) W satisfies the differential equation $\Delta W + k^2 W = 0$. b) W vanishes at the points of S. c) W is finite in all space except at a point P, the "luminous point," where W becomes infinite as $1/r$. d) W vanishes at infinity. We denote the constant k as the "reciprocal wavelength," even though the word wavelength has a direct meaning only for the special case of plane waves. The same conditions apply to the function V; U is then determined by the incompressibility condition.

We speak of a *Riemann double-space*. Namely, we imagine three-dimensional space as made of two realizations, both of which we cut along the surface S and join alternatively. The boundary curve C forms a branch curve for this double-space. We assume that there is function u which is single-valued in this double-space, satisfies the differential equation $\Delta u + k^2 u = 0$, vanishes at infinity, and has a pole at the one point P of the double-space, while it remains finite at the location $\overline{P}$ in the other realization of space. We denote this function as the *principal solution in the double-space*. To make clear the dependence on the position of the pole, we may write it as $u(P)$.

We now construct the mirror point P' of P with respect to the plane of S — this point falls in the second realization of space if P falls in the first — and form the difference $u' = u(P) - u(P')$. This difference satisfies all the conditions which are posed for V and W. For it becomes infinite in the first realization of space only at the location P, and it vanishes at points of S by considerations of symmetry. It is evident that the conditions a) and d) are fulfilled. It follows that the functions V and W become identical with the thus formed function u', and become single-valued by analytic continuation in the double-space just as u' does. One remarks, incidentally, that u' does not become zero simultaneously at points A of the Y, Z plane which lie outside C in the first realization of space. Namely, the path from P' to A is, as it were, longer than the path from P to A, since one can arrive from P' to A only through the interior of C. Only the points of the Y, Z plane inside C, not the points outside C, lie symmetrically between P and P'.

If we wish to have an exact solution of a diffraction problem, then we must begin with the formation of the principal solution $u(P)$. Our procedure can be characterized briefly as follows: *since there is no place in ordinary space for the mirroring process, we contruct a double-space, in which the mirroring process is possible.*

In the formation of the principal solutions, one will begin with the simplest case, where the light motion is independent of one coordinate (z), and the screen S has, correspondingly, a special form. The *Riemann space* reduces to a *Riemann surface*; of the *branch curve* C only the intersection points with the X, Y plane come into consideration; these become *branch points* in the usual sense. The differential equation simplifies to

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0.$$

As we see, we do not avoid even in optics the solution of *boundary value problems*, the general problems of mathematical physics. In the usual treatment of optics one proceeds, as is well known, quite differently.

One derives only a small number of functions from the nature of light (that is, from the differential equations of optics), and reduces more general problems to these functions by the so-called geometric method. This procedure has its basis in the smallness of the wavelength. In fact, if k becomes infinitely large, the differential equations of optics lose their meaning, and cease to represent conditions through which the magnitude of the light vector at a point is determined by its behavior at other points. It can be shown through a limiting process that light can then be decomposed into individual rays which propagate independently with constant velocity. One then arrives at a conception which corresponds, as it were, to the viewpoint of the *emission theory*, where one assumes that light particles are hurled forward without exerting interaction forces on each other. *While the undulation theory has been generally adopted theoretically, the actual treatment of optical problems remains fixed in many respects to the standpoint of the emission theory.*

From this standpoint, diffraction phenomena are impossible from the start. Nevertheless, such phenomena are derived on the basis of Huygens' principle. This principle is completely exact in itself, but its application to diffraction theory is not exact, even in the somewhat sharpened method of *Kirchhoff*. To calculate, for example, the component U according to Huygens' principle, the values of U and $\frac{\partial U}{\partial n}$ must be known on the diffracting screen. One now assumes these values as they would be given for infinitely small wavelength, and thus chooses U and $\frac{\partial U}{\partial n}$ on the illuminated side of the screen as if the screen were not present, and sets $U = 0$ and $\frac{\partial U}{\partial n} = 0$ on the opposite side. In the final results, however, one regards the wavelength as finite. The above boundary

conditions are absolutely inadmissible for finite wavelength. For they contain discontinuities, which in the derivation of Huygens' principle are explicitly excluded; further, it follows from $U = 0$ and $\frac{\partial U}{\partial n} = 0$, according to a known theorem, that U must vanish identically in all space. It is very curious that the formulas agree with observations so well in spite of these serious objections.

The usual procedure is abandoned for the first time in a work of Mr. *Poincaré*. Here the diffraction problem is conceived as what it is, namely as a boundary value problem. But even here the problem is carried out only for infinitely small wavelength.

The actual formation of the principal solution appears to involve great difficulties in complicated cases. The *mapping method*, through which the corresponding functions for the equation $\Delta u = 0$ can be easily derived, fails for our differential equation. On the other hand, the *method of power series* remains applicable here with the difference that the powers are replaced by Bessel functions. Namely, at an ordinary point of the Riemann surface we have

$$\sum J_n(kr)(A_n \cos n\varphi + B_n \sin \varphi),$$

and at a $(m-1)$-fold winding point

$$\sum J_{n/m}(kr)(A_n \cos \frac{n}{m}\varphi + B_n \sin \frac{n}{m}\varphi).$$

These series converge exactly as the corresponding series

$$\sum r^n(A_n \cos n\varphi + B_n \sin \varphi)$$

and

$$\sum r^{n/m}(A_n \cos \frac{n}{m}\varphi + B_n \sin \frac{n}{m}\varphi)$$

of potential theory, to the approach of the next singular point.

The simplest case of a Riemann surface with only one winding point in the finite region can be fully completed. The series here are only a point of departure; the result can be brought into the convenient form of a definite integral. Here I also treat a problem that for Mr. Poincaré remains uncompleted, namely the *diffraction by a screen of finite absorptivity*.

The calculations are carried out to comparison with observations. A very happy coincidence occurs. The formulas of the usual diffraction theory appear as more or less good approximations to our exact formulas. At the same time the formulas of Mr. Poincaré show, in spite of the difference in the starting point, a remarkable similarity with ours.

Index

absorptivity, 154
acoustics, 9, 49, 151
addition theorem, 48, 49
algebraic function, 12
analytic continuation, 11, 46, 62
angle of diffraction, 65, 66, 125, 131–133
Arago, D., 5
Aristotle, 148
Armin, H., 8

Baker, B. B., 5, 112, 124, 135
Berry, M., 3, 113, 135
Bessel function, 13, 14, 17–19, 22, 27–29, 31, 32, 38, 42, 43, 46–49, 78, 87, 120
Birch, T., 135, 142
black body, 124
Bologna, 3
Boltzmann, L., 121
Bonn, 133
Born, M., 6, 112, 113, 135
Boyle, R., 4
branch curve, 12, 152, 153
branch cut, 29, 58, 69, 71, 82, 84, 88, 89
branch line, 11
branch point, 17, 25, 28, 29, 44, 58, 66, 82, 83, 88, 153
Braunbek, W., 3, 135
Brougham, Lord, 4

Carslaw, H. S., 7, 69, 135
Cauchy's theorem, 24, 34, 35, 41, 45
Cavendish, H., 4
Chun, C., 2
Clemmow, P. C., 109, 135
conformal map, 34, 73, 74, 98
Copson, E. T., 5, 112, 124, 135
Courant, R., 78, 87, 136

D-line, 52, 60, 104
Dahn, F., 2
de la Hire, P., 148
Dechales, C., 136, 147
Descartes, R., 141, 148
Dodgson, C., 78
Donkin, W. F., 78

electrical waves, 151
electrostatics, 69, 71–76, 97, 98, 100, 101
Elster, L., 2
emission theory, 153
essential singularity, 13
ether, 4

Feshbach, H., 121, 137
Forman, P., 8, 136
Fourier series, 13, 14, 16, 19, 21
Fraunhofer, J., 128
Fresnel, A., 5, 9, 128, 133

fringes, 3–5, 131, 132

Göttingen, 2, 7, 9, 66
geometric optics, 2, 59, 61, 62
Gouy, L., 66, 113
Gradshteyn, I. S., 94, 136
grating, 66, 128
Grimaldi, F., 3, 136, 141, 142, 144–148

Hankel function, 78, 97
Hankel, H., 32, 42, 43, 136
heat conduction, 6
Heine function, 21
Heine, H., 18, 19, 28, 29, 32, 38, 48, 136
Helmholtz, H., 5
Herant, M., 115
Hermann, A., 136
Hertz, H., 121
Hertzian oscillations, 9, 49
Hilbert, D., 1, 78, 87, 136
Hooke, R., 3, 136, 142, 144, 147
Hurwitz, A., 1
Huygens' principle, 5, 153, 154
Huygens, C., 3, 5

incompressibility condition, 10, 152
inflection, 141, 148

Jackson, J. D., 124, 136

Königsberg, 1, 2, 6, 133
Kirchhoff, G., 5, 7, 9, 63–66, 121–128, 131–133, 136, 153
Klein, F., 2, 23
knife edge, 4, 133

Lamé function, 23
Lamb, H., 7, 136
Laukien, G., 3, 135
Laurent theorem, 22
Legendre polynomial, 87
Liebisch, A., 2
Liebisch, T., 2
light vector, 9
Lindemann, C., 1
Lossen, W., 2
luminous line, 4, 12, 61, 62, 133
luminous point, 6, 9–12, 43, 46, 61, 122, 123, 126–129, 132, 152

Maey, E., 66, 133, 136
Maskelyne, N., 4
Maxwell's equations, 4, 6, 58, 110, 111, 113, 122
Maxwell, J. C., 4, 23, 78, 137
Meixner, J., 113
Meyer, C. F., 5, 137
Morse, P., 120, 121, 137
Munich, 128

Neumann, C., 17, 18, 137
Newton, I., 3, 4, 141, 142
nodal line, 54, 106–109
null curve, 54–57, 107, 109

Oberhettinger, F., 137
optical intensity, 64, 65, 120, 126, 128, 130–132

Pape, C., 2
Pauli, W., 104, 137
Peacock, G., 4, 137
Peter, H., 4
Planck, M., 121
Pockels, F., 13, 16, 17, 23, 42, 137

Poincaré, H., 6, 7, 9, 58, 63, 66, 113–116, 118–121, 124, 137, 154
potential theory, 13
power series, 13–15, 19, 22, 37, 46, 47, 154
Priestley, J., 3, 137, 141, 149
Prussia, 1

Rahts, J., 2
Rayleigh, Lord, 8
Remscheid, 133
Riemann double-space, 6, 11, 69, 70, 152
Riemann surface, 6, 12, 13, 33–37, 43, 46, 57, 64, 66, 71–76, 82–84, 88, 89, 94–101, 106, 108, 109, 115, 151, 153, 154
Rohault, J., 148
Ryzhik, I. M., 94, 136

Saalschütz, L., 2
semiconvergent series, 42, 43, 51
shadow, 59–63, 65, 66, 113, 143–146
slit, 66
Sommerfeld (Mathiass), C., 1, 2
Sommerfeld, F., 1
spectroscopy, 121, 128
spherical function, 1, 22–25, 27–30, 40, 49, 87
  tesseral, 27, 29
  zonal, 27, 28, 87
Springer, G., 82, 138
stereographic projection, 22, 25, 40, 41, 78, 79, 82
Stratton, J. A., 112, 138
Strutt, J. W., 8, 139

Taylor series, 17, 19
Thiele, G., 2
transition strip, 52–57, 60, 62–64, 104, 107, 119, 120

undulation theory, 153

Voigt, W., 7, 139
Volkmann, P., 2

Walter, J., 2
Watson, G. N., 78, 87, 92, 139
wedge, 37, 46, 94, 97, 98, 100–102
Whittaker, E. T., 78, 139
Wiechert, E., 2
winding point, 7, 14, 17, 19, 33, 34, 37, 43, 49, 52, 54–57, 60–62, 65, 74, 115
winding surface, 33, 38, 39
Wolf, E., 6, 112, 113, 135

Young, T., 3, 4, 139